A Guided Tour Through Physical Oceanography

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Il libro della natura è scritto in lingua matematica

(The book of nature is written in the language of mathematics)

Galileo Galilei

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Chapter 1

Preface

This course deals with the aspects of physical oceanography which are important to understand the dynamics and role of the ocean in the climate system of our planet earth. There are outstanding and very comprehensive books on the dynamics of the worlds ocean, and the purpose of this *Guided Tour Through Physical Oceanography* is not to rival them, but rather to provide for a concise, self contained and systematic introduction to the field, emphasizing the basic questions. While teaching an introductory class of physical oceanography to graduate students, I found no concise introduction to the subject that deals with the matter on an advanced and modern level. By modern I mean oriented "[...] toward the understanding of physical processes which control the hydrodynamics of oceanic circulation." (H. Stommel, *The Gulfstream*, 1958). More precisely, when considering such physical process we first assume that such process can be modeled by the Navier-Stokes equations, an assumption that is surely satisfied to a very high degree of accuracy. We then proceed in four steps: (i) formulate assumptions about the process that simplify the problem; (ii) use these assumptions to derive simplified (mathematical) models; (iii) study the thus obtained simplified models; and (iv) compare the results to observations (if available) to validate or reject the results. If the results have to be rejected when confronted with observations, laboratory experiments or results from more complete models, we have to formulate new assumptions, that is, restart with (i). It is the first point, the choice of the important assumptions which is key to scientific progress and asks for a deep scientific insight.

Today's research on ocean dynamics is guided by the power of increasingly complex numerical models. These models are, however, so involved, that simpler models are needed to comprehend them and the study of a phenomenon of ocean dynamics passes by the study of a hierarchy of models of increasing complexity.

The prerequisites for this guided tour is a course in calculus and some knowledge of elementary fluid dynamics. I try to present the subject "as simple as possible but not simpler" (A. Einstein). It is indeed my conviction that some introductory courses of ocean dynamics are over simplified and are thus impossible to really understand or are plainly wrong.

Many important aspects of the ocean circulation are omitted, which is permitted in a *guided tour* but not so much in a text book. The most important are waves (surface, Poincaré, Kelvin and Rossby), which are not visited by this *guided tour*. The justification that we are here mostly concerned with the behavior of the ocean on long time scales, relevant to climate dynamics, much longer than the typical time scale of the above mentioned waves, is weak. Including oceanic waves in a self contained and systematic way would easily double the length of this course.

I choose not to present figures of observations and data in this course as they are subject to rapid improvement and as their latest version can easily be retrieved from the Internet. The search for such data and figures are given as exercises.

Chapter 2 is a short account of the observations of the worlds oceans in the past and present. In Chapter 3 we discuss the composition and thermodynamic properties of oceanic water. The forces acting on the ocean are discussed in Chapter 4. Chapter 5 is the key part of this *guided tour through physical oceanography*, where the basic concepts of oceanic circulation theory are derived. In chapter 6 we will see how the forcing put the water masses of the ocean into motion. An important question is: how can the forcing, which acts on the surface of the ocean influence the motion in the deep ocean? Indeed, over 250 years ago the famous mathematician L. Euler wrote: "La raison nous assure, et l'expérience nous confirme, que les courants pénétrent rarement jusqu fond de la mer." (Recherche sur la découverte des courants de mer, Leonhard Euler). (Reason insures us and experience confirms, that the currents penetrate rarely to the ocean floor (my translation)), and we will see which processes are responsible for generating and sustaining ocean currents down to the very bottom of the ocean.

Chapter 7 is dedicated to baroclinc phenomena, dynamics that can not be described by a single horizontal layer.

The overturning or thermohaline circulation, discussed in chapter 9, deals with the sinking of dense water masses in the high latitudes, their circulation in the abyssal ocean and their upward motion in the worlds ocean. This thermohaline circulation is important for the climate dynamics as it transports large amounts of heat, carbon dioxide and as it varies on climatic (long) time scales.

At the equator the Coriolis parameter, representing the effect of terrestrial rotation, vanishes. Chapter 8 discusses the consequences for the dynamics of the equatorial ocean.

This guided tour would not be complete without discussing the, small scale, three dimensional turbulent fluxes in the ocean. To model this processes the quasi two dimensional models used in the previous chapters are no longer adapted and different models, based on different assumptions have to be derived from the Navier-Stokes equations, this is accomplished in chapter 10.

Chapter 2

Observing the Ocean

A systematic determination of the bathimetry (depth structure) of the world ocean and its observation started with the *HMS Challenger* expedition (1872–1876). Besides biological and geological observations, the temperature was measured at different depths and locations of the worlds ocean and water samples were taken which were then analysed to determine the salinity and the composition of seawater. Current measurements in the open sea were more difficult to perform from a ship subject to current and wind-forces and could only be estimated.

Today the ocean is observed from research vessels which take measurements along a well defined trajectory and at given depth, from moorings which are attached to the ocean floor and take observations at one location during a certain period of time, usually a few years, and from floating buoys which are transported by the current but change their depth following a predefined schedule and communicate the measurements by satellite. These devices measure the velocity, temperature and chemical and biological composites of the oceanic water and provide us with a spatio-temporal, that is, a four dimensional picture of the dynamics and the composition of the worlds ocean. This picture is, however very patchy. At every instance in time large areas of the ocean go unobserved. Starting from the 1980's satellite observations measure the height of the sea surface, the sea surface temperature, the sea surface salinity and the ocean color of the sea surface at a spacial and temporal density and continuity unknown from previous observations. Satellites can, however, only provide us with data from the sea surface as electro-magnetic waves do not penetrate into the deep ocean. Much of today's knowledge of the worlds oceans is due to satellite observations and many efforts go to extrapolating this surface measurements into the deep ocean.

2.1 Geometry of the Ocean

The world ocean has a surface of $361 \times 10^6\text{km}^2$, and an average depth of 3.8km. The average depth is approximately the same in the Pacific, Atlantic and Indian Ocean.

Exercise 1: Search the Internet for maps of the bathimetry (depth) of the worlds ocean.

2.2 Variables measured

Using oceanic currents to accelerate and facilitate sea voyages is a concept as old as navigation itself. On the open sea ocean currents were however hard to detect. Sea men using the gulf stream to travel from the east coast of the US, realised quiet early that the gulf-stream water was warmer than the surrounding waters and they used temperature measurements to determine their position with respect to the gulf stream. During the cold war, Russian submarines used the same trick to approach the east coast of the US along the northern border of the Gulf Stream, where they could hide due strong density gradients and turbulent ocean dynamics deflecting the sonar signal.

Scientists were then interested in measuring other properties of sea water to determine the path ways of water masses in the ocean. Salinity measurements are easy to perform due to the strong relation to the electric conductivity of the sea water. Today a variety of constituents of the oceanic water masses are measured including: oxygen, carbon-dioxide, freon, radioactive tracers and the concentration of biological constituents. Freon gases, were released to the atmosphere starting from the first part of the 20th century and was stopped when it was found that they are responsible for the destruction of the atmospheric ozone layer, are dissolved in the ocean. By measuring their concentration in oceanic waters the age of the water masses, that is the time since their last contact with the atmosphere, can be obtained. The same applies to some radioactive traces which were released to the atmosphere during atomic bomb explosions in the atmosphere during the mid twentieth century.

Tracers as temperature and salinity which change the density of the water masses are called *active tracers* as they act on the dynamics through their buoyancy, tracers that have no substantial impact on density of the water mass and thus on the dynamics are named *passive tracers* . The measurement of passive tracers is nevertheless important as they provide information about the movement of water masses.

Exercise 2: Search the Internet for maps of sea surface temperature (SST) and sea surface salinity (SSS).

Chapter 3

Physical properties of sea water

Sea water has many physical properties: temperature, salinity, pressure, density, electric conductivity, thermal conductivity, viscosity, diffusivity of temperature, diffusivity of salt, compressibility, thermal expansion, thermal capacity, speed of sound, optical refraction index and many more. If we like to characterize a probe of sea water we do not have to measure all of these quantities as they are not all independent. Indeed thermodynamics teaches us that sea water is described by only three independent variables[1]. That means, if we have measured three of this properties say temperature, salinity and pressure all the other variables can be calculated (or looked up in a table) and do not have to be measured. The best known relation between physical properties is the function that allows to calculate the density of sea water from temperature, salinity and pressure it is called the *equation of state* .

$$\rho = \rho(T, S, P). \tag{3.1}$$

Density, or more precisely density differences, are of primary importance as they act due to the earths gravitational force on the dynamics of the ocean and is a primary source of motion in the ocean. We will thus further investigate the four properties appearing in the equation of state.

3.1 Salinity

Salinity is the easiest to comprehend, its concentration is given in grams of salt dissolved in one kilogram of sea water and is measured in practical salinity units (psu). If one kilogram of sea water contains 34.7 grams of salt, the sea water has a salinity of 34.7 psu. Since the 1980s this is not exactly true as salinity is determined by the conductivity of the water sample: the mass of dissolved salt in 1kg of sea water is actually around 1.005g times the salinity, depending on pressure and temperature. Typical values of salinity in the world ocean range from 33 to 37 psu. In marginal seas they differ from these typical values as these basins are often shallow and have higher fresh water fluxes per volume. In the Mediterranean Sea (average depth of 1500m) they vary between 37 and 39 psu, in the Red Sea (average depth of 490m) they typically measure between 40 and 42 psu, while in the Baltic Sea (average depth of 55m) they range from 10 to 20 psu. Marginal basins play an important role in the global ocean dynamics due to their role as "factories" of extreme water mass properties (salinity and temperature).

The sea salt is composed of different sorts of salt, although the salinity varies in the world ocean the ratio of the different salts is rather constant, an observation called *Dittmar's law*,

[1] We neglect the influence of: dissolved gases, chemical substances other than salt, variations in the composition of the sea salt and biology.

named after William Dittmar who, in 1884, analysed the waters collected by the scientific expedition of the British corvette, HMS Challenger (1872–1876). The major constituents of sea salt are shown in table 3.1. Small regional variations of the composition of sea salt are however present in the ocean and will probably to be included in the determination of a futur equation of state with a higher degree of accuracy.

Salt	percentage
Chloride	54
Sodium	31
Sulfate	8
Magnesium	4
Calcium	1
Potassium	1
others	1

Table 3.1: Major constituents of sea salt

3.2 Temperature and Potential Temperature

Temperature is measured in degrees Celsius (^{o}C) and temperature differences in Kelvin (K), oceanographers are however slow in adapting to the SI unit Kelvin to measure temperature differences.

The temperature of the world ocean typically ranges from -2^oC (-1.87^oC freezing point for $S = 35$ at surface) (freezing temperature of sea water) to 32^oC. About 75% of the world ocean volume has a temperature below 4^{o}C. Before the opening of the Drake Passage 30 million years ago due to continental drift, the mean temperature of the world ocean was much higher. The temperature difference in the equatorial ocean between surface and bottom waters was about 7K compared to the present value of 26K. The temperature in the Mediterranean Sea is above 12^oC even at the bottom and in the Red Sea it is above 20^{o}C.

If one takes a mass of water at the surface and descends it adiabatically (without exchanging heat with the environment) its *in situ* (latin for: in position; the temperature you actually measure if you put a thermometer in the position) temperature will increase due to the increase of pressure. Indeed if you take a horizontal tube that is 5km long and filled with water of salinity $S = 35$psu and temperature $T = 0^o$C and put the tube to the vertical then the temperature in the tube will monotonically increase with depth reaching $T = 0.40^o$C at the bottom. To get rid of this temperature increase in measurements oceanographers often use *potential temperature* θ (measured in ^{o}C) that is the temperature of a the water mass when it is lifted adiabatically to the sea surface. It is always preferable to use potential temperature, rather than *in situ* temperature, as it is a conservative tracer (see section 3.7). Differences between temperature and potential temperature are small in the ocean < 1.5K, but can be important in the deep ocean where temperature differences are small.

3.3 θ-S Diagrams

If one mixes the mass m_1 (measured in kg) of sea water of salinity S_1 with the mass m_2 of sea water of salinity S_2 one obtains the mass $m_1 + m_2$ of sea water of salinity

$$S_3 = \frac{m_1 S_1 + m_2 S_2}{m_1 + m_2}. \tag{3.2}$$

This follows from the definition of the salinity and the mass conservation.

If one mixes the mass m_1 of sea water of temperature θ_1 with the mass m_2 of sea water of temperature θ_2 one obtains the mass $m_1 + m_2$ of sea water of temperature (see fig. 3.3)

$$\theta_3 = \frac{m_1 \theta_1 + m_2 \theta_2}{m_1 + m_2}. \tag{3.3}$$

The above is only strictly true if the heat capacity does not vary with temperature and salinity, which is approximatrely true if we restrict ourselves to oceanic values (errors are typically smaller than 1%), and when the (negligible) heat of mixing is neglected.

The analysis of water masses are performed with the help of θ-S *diagrams* as shown in fig. 3.3

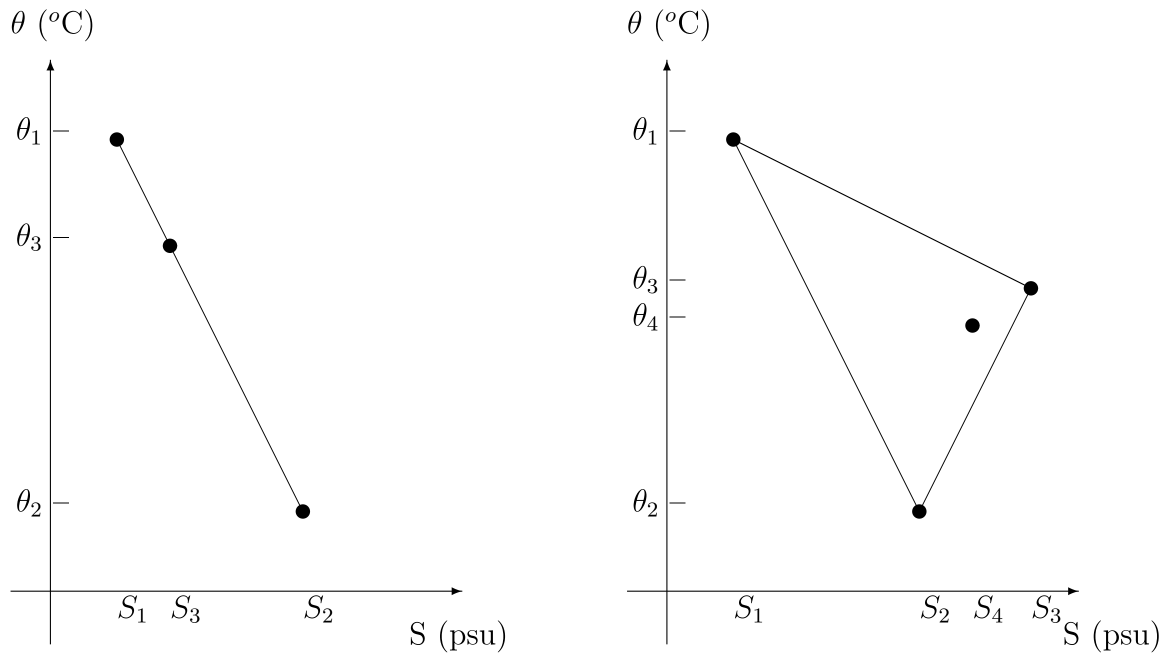


Figure 3.1: θ-S–diagram. Left: mixing of two water masses, the mixture of two water masses lies on a line between water masses. Right: mixing of three water masses, the mixture of 3 water masses lies within the triangle formed by the three water masses. The exact location can be obtained by eqs. 3.2 and 3.3.

3.4 Pressure

Pressure is measured in Pascal (1 Pa = 1 N m^{-2}). When pressure is considered, oceanographers usually mean hydrostatic pressure:

$$p(x, y, z) = p_{\text{atmos}} + g \int_z^0 \rho(z') dz', \tag{3.4}$$

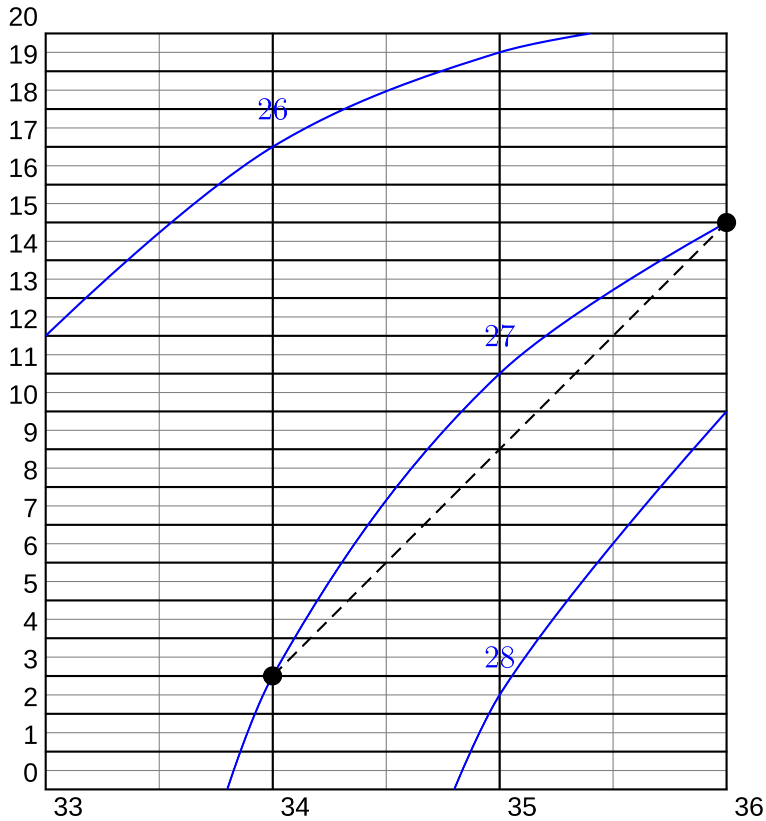


Figure 3.2: θ-S-diagram, potential density 0 dbar lines and σ-values are shown in blue. It is clearly seen that the mixture of two watermasses (dots) of equal densty, which lies on the dashed line, is always denser than the initial water masses.

due to the atmospheric pressure p_{atmos} and the product of density ρ and gravitational acceleration of the overlaying fluid. Please note that also for oceanographers the upward direction is the positive direction, although they mostly speak of depth, this often leads to confusion. When using the hydrostatic pressure we neglect the usually small variations of pressure due to the fluid motion (acceleration of fluid). In the equations of motion (see 5.1 – 5.3), it is not the pressure, but its gradient that matters, which means, that only changes in pressure but not the absolute values are of importance to the dynamics. This allows oceanographers to furthermore neglect the atmospheric pressure and define that at the ocean surface $p = 0$. Other units of pressure are bars (1 bar $= 10^5$ Pa), or decibar (1 dbar $= 10^4$ Pa) which is roughly the increase of pressure when the ocean depth increases by 1m.

Attention: pressure is a scalar quantity, that is, has no direction!

3.5 Density and σ

Density is measured in kg/m^3 and typical values for sea water range from 1020 – 1050kg/m^3 the density of sea water is usually given in sigma-values $\sigma_T(T,S) = (\rho(T,S,0) - 1000\text{kg/m}^3)$ $/(1\text{kg/m}^3)$, that is a water of $\rho(10^o, 35, 0) = 1031.0\text{kg/m}^3$ has $\sigma(10^o, 35) = 31.0$ (no units!). The σ_T (sigma-sub-T) value refers to the density a water mass at temperature T and Salinity S has at the ocean surface.

Density depends on temperature, salinity and pressure in a non-linear way and these non linearities lead to many interesting phenomena. The actual dependence, for values typical to the ocean, is given by the *UNESCO 1981 formula* which is a best fit to laboratory measurements. A numerical version of this formula is implemented in all numerical models of the ocean dynamics.

The non linearity of the equation of state leads to interesting and important phenomena in oceanography. One is *cabbeling* which means that by mixing of two water masses the resultant water mass has a density which is superior than the weighted mean density. As shown in fig. 3.2 if the two water masses have the same density their mixture, somewhere on the dashed line, has a larger density.

It is the difference in density that is dynamically important. We have seen in section 3.2 that two water mmasses which are at different depth might have the same temperature but different potential temperature. As it is the potential temperature that is conserved by a water mass when move adiabatically it seems more natural to measure sigma values in terms of potential temperature. To compare densities of water masses oceanographers introduced the notion of potential density, where $\sigma_0(\theta, S)$ is the "sigma value" of a water mass of potential temperature θ and salinity S when brought adiabatically (no exchange of heat) to the sea surface. Potential density is, unfortunately, not the answer to all the problems, as two water masses which have the same σ_0 might have differnt densities at depth. This is again a consequence of the non-linearity in the state equation called *thermobaricity* which is due to the fact that warmer water is less compressible than colder water. This can be seen in fig. 3.3 where the sigma density of two water masses is given at the pressure of 0 dbar (at the surface) and at 4000 dbar. The water mass which is heavier at the surface is actually lighter at 4000 dbar. This lead to the definition of not only the potential density at the surface σ_0 but also for example to σ_{4000}, which gives the sigma value of a water parcel when transported adiabatically to a pressure of $4000 dbar \approx 4000$m depth.

Locally the dependence can be written:

$$\rho(T + \delta T, S + \delta S, p + \delta p) = \rho(T, S, p)(1 - \alpha\delta T + \beta\delta S + \gamma\delta p), \tag{3.5}$$

where α is the thermal expansion coefficient, β is the saline contraction coefficient and γ the compressibility of sea water. The non-linearity of the state equation arises from the fact that all these coefficients are themselves functions of temperature, salinity and pressure.

3.6 Heat Capacity

The dynamics of the ocean is important for our climate due to its transport of heat from the low to the high latitudes. The heat capacity of sea water is around 4.0×10^3 J (K kg)$^{-1}$, about four times the value of air. At the sea surface air is almost 770 times less dense than water. At equal volume water contains approximately 3000 times more heat than air.

Exercise 3: Suppose that the atmosphere above the ocean has a constant temperature (independent of height) and that the ocean underneath is at the same temperature. What is the depth of the ocean if it contains the same amount of heat as the atmosphere above? ($C_p(seawater) = 4.0 \times 10^3 J/(kgK)$ and $C_p(air) = 1.0 \times 10^3 J/(kgK)$). Do not use the thickness of the atmosphere in your calculations.

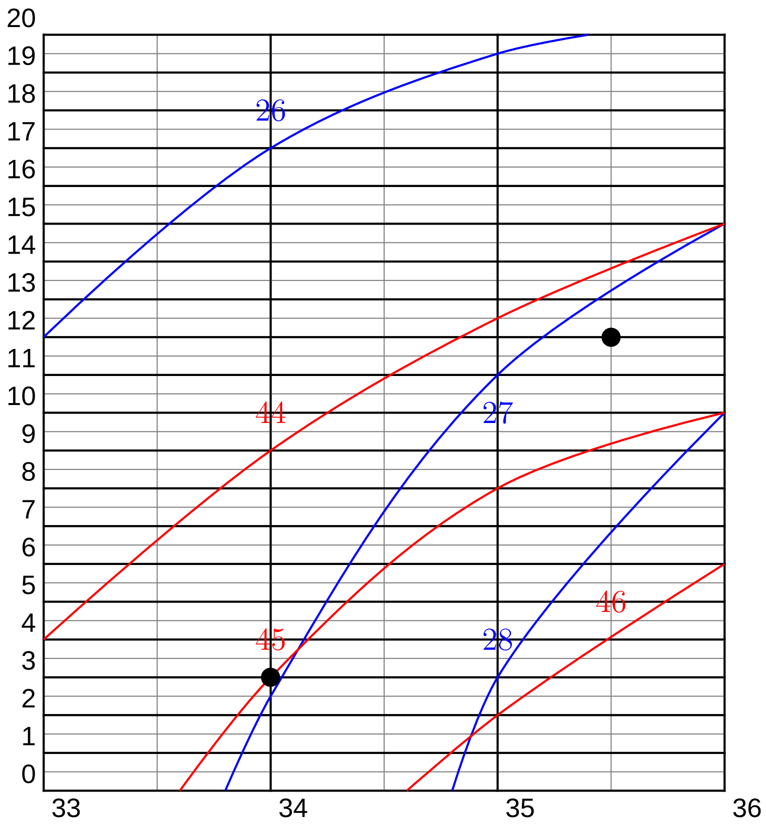


Figure 3.3: θ-S-diagram; potential density 0 dbar lines and σ_0-values are shown in blue; potential density 4000 dbar lines and σ_{4000}-values are shown in red. The figure ilustrates the phenomenon of thermobaricity. The hotter and saltier water mass is heavier at the surface and lighter at 4000m depth than the other water mass. This happens because hot water is less compressible than colder water.

3.7 Conservative Properties

The dynamics of a tracer S transported by an incompressible fluid of velocity field $\mathbf{u}$, diffused with the diffusion κ and subject to sources and sinks Q is govenred by the advection diffusion equation:

$$\partial_t S + \mathbf{u} \cdot \nabla S - \nabla \cdot (\kappa \nabla S) = Q. \tag{3.6}$$

A scalar is said to be conservative if $Q = 0$.

Besides their important influence on the density there is another reason why salinity and potential temperature play such an important role in oceanography they are conservative. Away from the boundaries these properties can only be changed by mixing with water masses of different characteristics. Please note, that temperature is not conservative as it changes when a water parcel is transported up or down in the ocean, when pressure changes. This is equivalent to say, that there are no sources or sinks of salinity and potential temperature in the ocean interior.

Other scalars like dissolved oxygene nutrients and biological concentrations are not conservative as they have sinks and sources.

3.8 Water Masses in the Ocean

Water in the ocean mixes due to molecular diffusion and turbulent stirring (see section 10.2). In large parts of the ocean, away from the boundaries, the mixing is small. Water parcels thus conserve their conservative characteristics, salinity and potential temperature, when transported over long distances and a mean large scale transport velocity of water masses and the velocity field in the ocean can be determined by measuring potential temperature and salinity. Further more, water mass characterisitics change only slowly in the deep ocean and show only small variations over the years. These changes can be used as important indicators of climate (long-time-large-space) variability.

3.9 Sea Ice and Ice Bergs

Fresh water freeszes at 0^oC and sea water with a salinity of 35psu freezes at -1.8^oC, at atmospheric pressure. Fresh water has its maximal density at 4^oC, when lakes cool below this temperature the cold water stays at the surface and freezing happens quickly near the surface while 4^oC warm water is found in the deep lake. The ice formation of lakes mostly depends on the atmospheric temperature and the wind speed. For water with a salinity over 24.7psu the maximum density is at the freezing temperature. When the ocean is cooled the cold surface water descents and is replaced by warmer water from depth until the freezing temperature is reached. Sea water can only freeze when the cooling from the atmosphere is stronger than the convective warming from the deep ocean. So, for the formation of sea ice, besides atmospheric temperature and the wind speed, the water depth and the stratification of the ocean in temperature and salinity are important parameters. No such convective warming is present for fresh water lakes once the temperature is below 4^oC, and indeed you can go ice-skating on lakes in northern Europe while the nearby sea is completely ice free. The vertical convection process is studied in section 10.3.

Ice cover are crucial to the ocean dynamics as it: (i) has a strong influence on the reflection of the incoming radiation, especially when they are covered by fresh snow (see section 4.1.1),

(ii) reduces the transfert of heat, isolating the ocean, (iii) actes as a thermal resevoir due to the latent heat associated with melting and freezing and (iv) changes the salinity and buoyancy, through melting and freezing. The first point is responsible for the fact that polar regions are very sensitive to global climate change. A little change in temperature can freeze or melt sea ice, the positive feed back of the albedo will then amplify the initial temperature change.

The Arctic Ocean is almost completely surrounded by land, while the Antarctic Ocean is completely open towards lower latitude. Arctic sea ice is thus hindered to travel towards lower latitude and typically survives several summer periods and has a typical thickness from two to three meters. Only about 10 % of the arctic ice travels south through the Fram Strait every year. Whereas the Antarctic ice is mostly seasonal, with 80 % disappearing by the end of the austral summer and has a typical thickness from one to two meters.

During freezing salt becomes trapped in the ice forming brine pockets which have a size around 10^{-4}m. The amount of salt trapped in the ice during freezing increases with the growth rate of the sea ice and the salinity of the seawater. Newly formed ice has a typical salinity around 14psu which is roughly half of the salinity of seawater. Within the sea ice the brine moves downward and leaves the ice at its lower boundary. The overall salinity of sea ice decreases with its age, leading to different salinities of the seasonal ice in the Antarctic and the multi-year ice in the Arctic. The salinity of ice has important influence on its thermal properties such as heat capacity, thermal conductivity and latent heat content. Sea ice also contains air bubbles, their volume typically icreases with age, reducing the ice density. A typical value for air bubble volume of multi-year ice is 15%. Sea ice is a multi-component multi-phase material. The fraction of each component and phase is subject to change due to exterior forcings.

Ice bergs are broken off (calved) parts of land glaciers and ice shelves, land glaciers that have migrated into the ocean, and do not contain significant amounts of salt. Ice bergs calved from ice shelves can have a horizontal extent of 100km.

Chapter 4

Surface fluxes, the forcing of the ocean

The principal source of the ocean dynamics are the fluxes through the ocean surface. The principal fluxes at the surface of the ocean are:

- heat flux
- fresh water flux
- momentum flux
- other chemical fluxes

The major source of ocean currents is the momentum flux provided by the wind-shear at the ocean surface. The first two of these fluxes, provided by heating-cooling and precipitation-evaporation at the ocean surface, create density differences influencing the ocean dynamics. The primary source of all these energy fluxes is the sun.

4.1 Heat Flux

The heat flux can be decomposed in four major contributions:

$$Q = Q_{\text{shortwave}} + Q_{\text{infrared}} + Q_{\text{sensible}} + Q_{\text{latent}} + \epsilon \tag{4.1}$$

Where we define $Q > 0$ if the ocean receives energy.

The first two are radiative fluxes which will be discussed in the next subsection, followed by sections discussing the sensible heat fluxes due to molecular exchange of heat and latent heat fluxes due to evaporation (and condensation of moist air).

4.1.1 Radiative Fluxes

The wave length of irradiation of a black body depends on its temperature (law of Wien). The radiation from the sun which has an average temperature around 6000K has a wave length around $\lambda = 0.48\mu m$ (short wave) ($1\mu m = 10^{-6}m$), which is in the short wave (visible) spectra. The radiation of the ocean and atmosphere, with an average surface temperature of 283K has a wave length around $\lambda = 10\mu m$, which is in the infrared spectra. The energy radiated is proportional to the fourth-power of the temperature (law of Stefan-Boltzmann)

The short wave heat flux of the sun just above the atmosphere is given by the solar-(NON)-constant which has an average value of 1.37kW m^{-2}, and varies $\pm 4\%$. Roughly two thirds of this variability is due to the varying distance between the sun and the earth and the reset due to the variability of the solar radiation. The solar irradiance has a marked cycle of 11 years leading to a variabilty of only 0.1%. The effects of (small) shifts in the frequency of the solar irradiance on the earths heat budget are unexplored. Due to geometry of the earth, its inclination of its rotation axis to the sun, and its rotation, the short wave heat flux at the top of the atmosphere during one day is a function of the day in the year and the latitude.

Part of the incoming short wave radiation is reflected back to space by the atmosphere, the amount strongly depends on the cloudiness.

The quantity of short wave heat energy absorbed by the earth depends also on the extent to which it reflects radiation. This is measured by the albedo which is the ratio of reflected energy to incident energy. The albedo is a dimensionless number between zero for a black body which absorbs all radiation, and one, for an object reflecting all light as, for example, a perfect mirror. Various values of albedo are given in table. 4.1.1.

Surface type	albedo
fresh snow	0.7 – 0.9
ice	0.3 – 0.4
ocean surface	0.05 – 0.15

Table 4.1: Albedo

The albedo of the ocean depends also on the roughness of the ocean (waves) and the angle of incidence of the radiation. The errors in the determination of the albedo of the sea surface present a major source of error in the estimation of the heat uptake of the ocean.

Exercise 4: Search the Internet for typical values of the albedo of the ocean.

Exercise 5: Search the Internet for a global map of the short-wave and long-wave radiative heat fluxes.

4.1.2 Sensible Heat Flux

The sensible heat flux, that is the heat fluxes between the ocean and the atmosphere due to molecular exchange of heat (not matter) is mostly negative, as the ocean surface, which is heated by the radiation of the sun, is on average warmer than the atmosphere just above. The sensible heat flux depends on the density of air $\rho_{\text{air}} = 1.3\text{kg m}^{-3}$ its heat capacity $C_p = 1.3\times10^3\text{J (K kg)}^{-1}$, the wind speed usually measured at the reference level of 10 meters above the ocean surface $|\mathbf{u_{10}}|$ and the local temperature difference between the sea surface temperature (SST) and the atmosphere 10 meters above the ocean.

The bulk formula to calculate the sensible heat flux is

$$Q_{\text{sensible}} = \rho_{\text{air}} C_p C_s |\mathbf{u_{10}}| (T_{\text{atmos}} - SST), \tag{4.2}$$

where the sensible heat transfer coefficient $C_s = 900$ is empirically determined.

Exercise 6: Search the Internet for a map of the sensible heat flux of the world ocean.

4.1.3 Latent Heat Flux

Evaporation is the major loss of heat by the ocean. The most energetic water molecules have enough energy to escape the water reducing this way the average temperature of the remaining water. The thus produced latent heat flux (which is almost always negative; except in the rare cases when hot moist air overlies the ocean and energetic fluid molecules enter the ocean) depends on the latent heat coefficient $L_E = 2.5 \times 10^6$J kg^{-1}, the wind speed measured at the reference level 10 meters above the ocean surface $|\mathbf{u_{10}}|$ and the relative humidity of the atmosphere 10m above the ocean q_a which is measured in kg of water vapour by kg of air and q_s is the saturation value which is a function of the sea surface temperature (SST), where it is supposed, that the air just above the ocean is saturated with water vapor.

The flux is approximated by a bulk formula:

$$Q_{\text{latent}} = \rho_{\text{air}} L_E C_L |\mathbf{u_{10}}|(q_s - q_a), \tag{4.3}$$

where the latent heat transfer coefficient $C_L = 1.35 \times 10^{-3}$ is empirically determined.

Exercise 7: Search the Internet for a map of the latent heat flux of the world ocean.

Exercise 8: Why is 10m chosen as the reference level?

4.1.4 Other Heat Fluxes

There are other sources of heat fluxes to the ocean which are however much smaller than the fluxes discussed above. Bio-chemical processes heat the ocean, as do naturally occurring radioactive processes, geothermal energy from the interior of our planet and internal friction of the fluid motion. Although the total energy fluxes of these processes are small they might be important locally in the ocean, an example are underwater volcanoes which heat up the ocean locally.

4.2 Fresh Water Flux

Fresh water fluxes are mainly due to rain, evaporation, condensation, melting of sea ice, freezing of sea water and river runoff. Recent research also suggests that a substantial amount of fresh water enters the oceans through ground water fluxes.

Exercise 9: Search the Internet for a map of the fresh water flux of the world ocean.

4.3 Wind Shear

The wind shear (a 2 dimensional vector quantity) is the major source of motion of the oceanic water masses. Many attempts have been made to obtain the exact shear as a function of the wind-velocity at the reference level of 10 meters above the ocean.

$$\tau = c_D \rho_{\text{air}} |\mathbf{u_{10}}| \mathbf{u_{10}} \tag{4.4}$$

Where the drag coefficient c_D is also a function of the wind velocity measured 10m above the sea surface $\mathbf{u_{10m}}$ and the density of air is around $\rho_{\text{air}} = 1.3$kg m^{-3}. The drag coefficient is often estimated to be $c_D = 1.3 \times 10^{-3}$. More recent research suggests:

$$c_D = (0.29 + \frac{3.1m/s}{u_{10m}} + \frac{7.7(m/s)^2}{u_{10m}^2})/1000 \text{ for } 3m/s < |u_{10m}| < 6m/s, \tag{4.5}$$

$$c_D = (0.6 + .070u_{10m})/1000 \text{ for } 6m/s < |u_{10m}| < 26m/s. \tag{4.6}$$

This empirical formulas can at best be seen as a sophisticated rule of thump. Note that the drag coefficient increases with wind speed. This is due to the larger roughness (waves) of the ocean surface with higher winds.

Uncertainties in the determination of the wind stress are a major difficulty in modelling the ocean dynamics.

Exercise 10: Search the Internet for a map of the wind shear of the world ocean.

4.4 What about tides?

The tidal dynamics varies on time scales that are very short compared to the large scale circulation relevant for climate issues and has no direct dominant influence on the long-term large-scale dynamics of the ocean. Tides do however substantially affect the large scale circulation by increasing the vertical mixing of the ocean due to the interaction of tidal motion and topographic features of the ocean basin. The tidal energy used to vertically mix the ocean is however difficult to estimate, as are the locations where this mixing occurs. These questions are actually an important subject of research in physical oceanography.

Chapter 5

Dynamics of the Ocean

5.1 From the Navier-Stokes to the Shallow Water Equations

The dynamics of and incompressible fluid is described by the Navier-Stokes equations:

$$\partial_t u + \quad u\partial_x u + v\partial_y u + w\partial_z u + \tfrac{1}{\rho_0}\partial_x P \quad = \nu\nabla^2 u \tag{5.1}$$

$$\partial_t v + \quad u\partial_x v + v\partial_y v + w\partial_z v + \tfrac{1}{\rho_0}\partial_y P \quad = \nu\nabla^2 v \tag{5.2}$$

$$\partial_t w + \quad u\partial_x w + v\partial_y w + w\partial_z w + \tfrac{1}{\rho_0}\partial_z P \quad = -g\frac{\rho}{\rho_0} + \nu\nabla^2 w \tag{5.3}$$

$$\partial_x u + \partial_y v + \partial_z w \quad = 0 \tag{5.4}$$

$$+ \quad \text{boundary conditions}$$

where u is the zonal, v the meridional and w the vertical (positive upward even in oceanography) velocity component, P the pressure, ρ density, ρ_0 the average density, ν viscosity of seawater, g gravity, and $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ is the Laplace operator.

The equation of a scalar transported by a fluid is:

$$\partial_t T + \quad u\partial_x T + v\partial_y T + w\partial_z T \quad = \kappa_T\nabla^2 T \tag{5.5}$$

$$+\text{boundary conditions} \tag{5.6}$$

$$\partial_t S + \quad u\partial_x S + v\partial_y S + w\partial_z S \quad = \kappa_S\nabla^2 S \tag{5.7}$$

$$+\text{boundary conditions} \quad ,$$

where T is temperature, S is salinity and κ_T, κ_S are the diffusivities of temperature and salinity. The state equation:

$$\rho = \rho(S, T, P) \tag{5.8}$$

allows to obtain the density from salinity, temperature and pressure.

The above equations describe the motion of the ocean to a very high degree of accuracy, but they are much too complicated to work with, even today's and tomorrows numerical ocean models are and will be based on more or less simplified versions of the above equations.

These equations are too complicated because:

- Large range of scales; from millimeter to thousands of kilometers
- Nonlinear interactions of scales

- How is pressure P determined, how does it act?
- Complicated boundary conditions; coastline, surface fluxes ...
- Complicated equation of state (UNESCO 1981)
-

A large part of physical oceanography is in effect dedicated to finding simplifications of the above equations. In this endeavour it is important to find a balance between simplicity and accuracy.

How can we simplify these equations? Two important observations:

- The ocean is very very flat: typical depth (H=4km) typical horizontal scale (L=10 000 km)
- Sea water has only small density differences $\Delta\rho/\rho \approx 3 \cdot 10^{-3}$

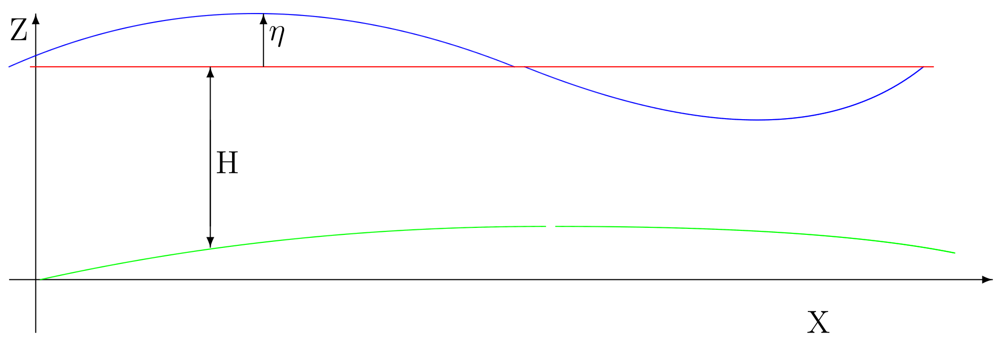


Figure 5.1: Shallow water configuration

Using this we will try to model the ocean as a *shallow homogeneous* layer of fluid, and see how our results compare to observations.

Using the shallowness, equation (5.4) suggests that w/H is of the same order as u_h/L, where $u_h = \sqrt{u^2 + v^2}$ is the horizontal speed, leading to $w \approx (Hu_h)/L$ and thus $w \ll u_h$. So that equation (5.3) reduces to $\partial_z P = -g\rho$ which is called the *hydrostatic approximation* as the vertical pressure gradient is now independent of the velocity in the fluid.

Using the homogeneity $\Delta\rho = 0$ further suggest that:

$$\partial_{xz} P = \partial_{yz} P = 0. \tag{5.9}$$

If we derive equations (5.1) and (5.2) with respect to the vertical direction we can see that if $\partial_z u = \partial_z v = 0$ at some time this propriety will be conserved such that u and v do not vary with depth. (We have neglected bottom friction). Putting all this together we obtain the following equations:

$$\partial_t u + \quad u\partial_x u + v\partial_y u + \tfrac{1}{\rho}\partial_x P \quad = \nu\nabla^2 u \tag{5.10}$$

$$\partial_t v + \quad u\partial_x v + v\partial_y v + \tfrac{1}{\rho}\partial_y P \quad = \nu\nabla^2 v \tag{5.11}$$

$$\partial_x u + \partial_y v + \partial_z w \quad = 0 \tag{5.12}$$

$$\text{with} \quad \partial_z u = \partial_z v = \partial_{zz} w \quad = 0 \tag{5.13}$$

$$+ \quad \text{boundary conditions}$$

What are those boundary conditions? Well on the ocean floor, which is supposed to vary only very slowly with the horizontal directions, the vertical velocity vanishes $w = 0$ and it varies linearly in the fluid interior (see eq. 5.13). The ocean has what we call a free surface with a height variation denoted by η. The movement of a fluid partical on the surface is governed by:

$$\frac{d_H}{dt}\eta = w(\eta) \tag{5.14}$$

where $\frac{d_H}{dt} = \partial_t + u\partial_x + v\partial_y$ is the horizontal Lagrangian derivation. We obtain:

$$\partial_t\eta + u\partial_x\eta + v\partial_y\eta - (H+\eta)\partial_z w = 0 \tag{5.15}$$

or

$$\partial_t\eta + u\partial_x(H+\eta) + v\partial_y(H+\eta) + (H+\eta)(\partial_x u + \partial_y v) = 0. \tag{5.16}$$

Using the hydrostatic approximation, the pressure at a depth d from the unperturbed free surface is given by: $P = g\rho(\eta + d)$, and the horizontal pressure gradient is related to the horizontal gradient of the free surface by:

$$\partial_x P = g\rho\partial_x\eta \text{ and } \partial_y P = g\rho\partial_y\eta \tag{5.17}$$

Some algebra now leads us to the shallow water equations (sweq):

$$\begin{aligned}
\partial_t u + & \quad u\partial_x u + v\partial_y u + g\partial_x\eta = \nu\nabla^2 u & (5.18)\\
\partial_t v + & \quad u\partial_x v + v\partial_y v + g\partial_y\eta = \nu\nabla^2 v & (5.19)\\
\partial_t\eta + & \quad \partial_x\left[(H+\eta)u\right] + \partial_y\left[(H+\eta)v\right] = 0 & (5.20)\\
& +\text{boundary conditions} .
\end{aligned}$$

All variables appearing in equations 5.18, 5.19 and 5.20 are independent of z!

5.2 The Linearized One Dimensional Shallow Water Equations

We will now push the simplifications even further, actually to its non-trivial limit, by considering the linearized one dimensional shallow water equations. If we suppose the dynamics to be independent of y and if we further suppose $v = 0$ and that H is constant, the shallow water equations can be written as:

$$\begin{aligned}
\partial_t u + & \quad u\partial_x u + g\partial_x\eta = \nu\nabla^2 u & (5.21)\\
\partial_t\eta + & \quad \partial_x\left[(H+\eta)u\right] = 0 & (5.22)\\
& + \text{boundary conditions.}
\end{aligned}$$

if we further suppose that $u^2 \ll g\eta$ that the viscosity $\nu \ll g\eta L/u$ and $\eta \ll H$ then:

$$\begin{aligned}
\partial_t u + g\partial_x\eta = 0 & \quad (5.23)\\
\partial_t\eta + H\partial_x u = 0 & \quad (5.24)\\
+\text{boundary conditions,}
\end{aligned}$$

which we combine to:

$$\partial_{tt}\eta = gH\partial_{xx}\eta \quad (5.25)$$
$$\text{+boundary conditions.}$$

This is a one dimensional linear non-dispersive wave equation. The general solution is given by:

$$\eta(x,t) = \eta_0^-(ct-x) + \eta_0^+(ct+x) \quad (5.26)$$
$$u(x,t) = \frac{c}{H}(\eta_0^-(ct-x) - \eta_0^+(ct+x)), \quad (5.27)$$

where η_0^- and η_0^+ are arbitrary functions of space only. The speed of the waves is given by $c = \sqrt{gH}$ and perturbations travel with speed in the positive or negative x direction. Note that c is the speed of the wave not of the fluid!

Rem.: If we choose $\eta_0^-(\tilde{x}) = \eta_0^+(-\tilde{x})$ then initially the perturbation has zero fluid speed, and is such only a perturbation of the sea surface! What happens next?

An application of such equation are Tsunamis if we take: $g = 10\text{m/s}^2$, $H = 4\text{km}$ and $\eta_0 = 1\text{m}$, we have a wave speed of $c = 200\text{m/s}= 720\text{km/h}$ and a fluid speed $u_0 = 0.05\text{m/s}$. What happens when H decreases? Why do wave crests arrive parallel to the beach? Why do waves break?

You see this simplest form of a fluid dynamic equation can be understood completely. It helps us to understand a variety of natural phenomena.

Exercise 11: Is it justified to neglect the nonlinear term in eq. (5.21) for the case of a Tsunami?

Exercise 12: Show that the linearized 1D SWE conserves energy when we define $E_{kin} = \int \frac{\rho H}{2} u^2 dx$ and $E_{pot} = \int \frac{\rho g}{2}\eta^2 dx$

Exercise 13: Show that the non-linear 1D SWE (with $\nu = 0$) conserves energy when we define $E_{kin} = \int \frac{\rho(H+\eta)}{2} u^2 dx$.

5.3 Reduced Gravity

Suppose that the layer of fluid (fluid 1) is lying on a denser layer of fluid (fluid 2) that is infinitely deep. $H_2 \to \infty \Rightarrow c_2 \to \infty$, that is perturbations travel with infinite speed. This implies that the lower fluid is always in equilibrium $\partial_x P = \partial_y P = 0$. The lower fluid layer is passive, does not act on the upper fluid but adapts to its dynamics, so that $\eta_1 = \frac{\rho_1-\rho_2}{\rho_1}\eta_2$. If we set $\eta = \eta_1 - \eta_2$ then $\eta = \frac{\rho_2}{\rho_2-\rho_1}\eta_1$ and the dynamics is described by the same sweqs. 5.18, 5.19 and 5.20 with gravity g replaced by the *reduced gravity* $g' = \frac{\rho_2-\rho_1}{\rho_2} g$ (“sw on the moon”).

Example: $g' = 3 \cdot 10^{-3} g$, $H = 300\text{m}$, $\eta_1 = .3\text{m}$ we get a wave speed $c = \sqrt{g'H} = 3\text{m/s}$ and a fluid speed of $u = 1\text{m/s}$.

Comment 1: when replacing $g\eta$ by $g'\eta$ it seems, that we are changing the momentum equations, but in fact the thickness equation is changed, as we are in the same time replacing the deviation of the free surface η (which is also the deviation of the layer thickness in not-reduced-gravity case) by the deviation of the layer thickness η , which is $(\rho_2 - \rho_1)/\rho_2$ times the surface elevation in the reduced gravity case. This means also that every property which is derived only from the momentum equations not using the thickness equation is independent of the reduced gravity.

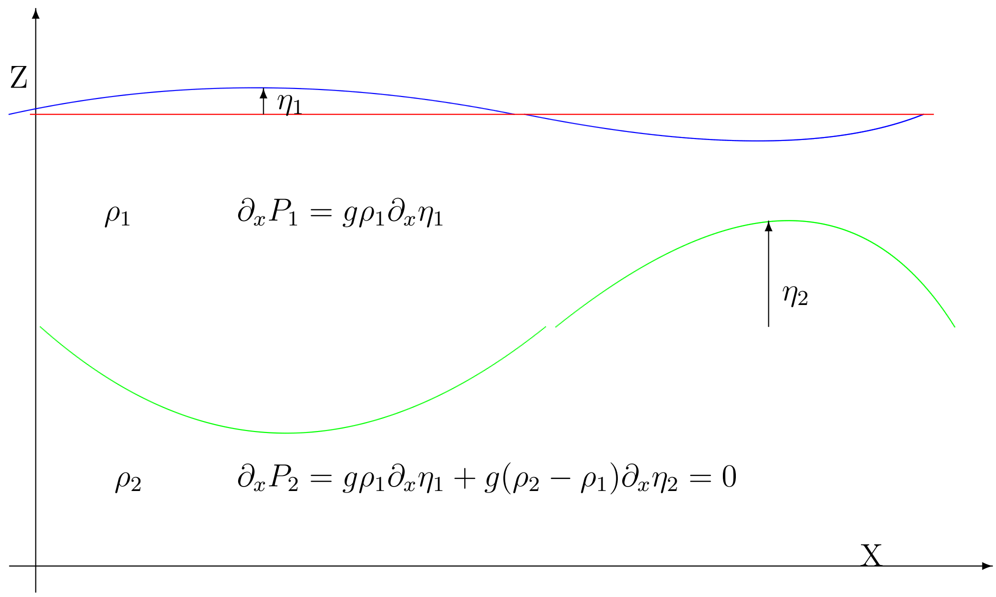


Figure 5.2: Reduced gravity shallow water configuration

Comment 2: fig. 5.3 demonstrates, that the layer thickness can be measured in two ways, by the deviation at the surface (η_1) or by density structure in the deep ocean (η_2). For ocean dynamics the surface deviation for important dynamical features, measuring hundreds of kilometers in the horizontal, is usually less than 1m whereas variations of (η_2) are usually several hundreds of meters. Historically the measurement of the density structure of the ocean to obtain η_2 are the major source of information about large scale ocean dynamics. Todays satellites measure the surface elevation of the ocean (altimetry) at a spatial and temporal density unknown before and are today our major source of information.

5.4 Vector Fields

5.4.1 Two Dimensional Flow

We have seen in the previous sections, that the dynamics of a shallow fluid layer can be described by the two components of the velocity vector $(u(x, y, t), v(x, y, t))$ and the surface elevation $\eta(x, y, t)$. The vertical velocity $w(x, y, t)$ is, in this case, determined by these 3 variables. The vertical velocities in a shallow fluid layer are usually smaller than their horizontal counterparts and we have to a good approximation a two dimensional flow field.

The dynamics of a fluid is described by scalar (density, pressure) and vector quantities (velocity). These quantities can be used to construct other scalar quantities (tensors, of order zero) vectors (tensors, of order one) and higher order tensors. Higher order tensors can than be contracted to form lower order tensors. An example is the velocity tensor (first order) which can be used to calculate the speed (its length), it is a tensor of order one. The most useful scalar quantities are those which do not change when measured in different coordinate systems, which might be translated by a given distance or rotated by a fixed angle. Such quantities are called *well defined* . For example it is more reasonable to consider the length of the velocity vector (the speed, well defined) rather than the first component of the velocity vector, which changes when the coordinate system is rotated. The length of the velocity vector is used to calculate the kinetic energy. The most prominent second order tensor is the strain tensor, considering the linear deformation of a fluid volume. In two dimension it is:

$$(\nabla \mathbf{u}^t)^t = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \tag{5.28}$$

if a coordinate system $\mathbf{e}'$ is rotated by an angle α with respect to the original coordinate system $\mathbf{e}$ the components in the new system are given by:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{A}\mathbf{u} \tag{5.29}$$

and $\mathbf{u} = \mathbf{A}^t\mathbf{u}'$. Note that $\mathbf{A}^t = \mathbf{A}^{-1}$.

For a scalar field (as for example temperature) we can express in the two coordinate systems: $f'(x', v') = f(x(x', v'), y(x', v'))$ we have:

$$\partial_{x'} f' = \frac{\partial f'}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial x'}\frac{\partial f}{\partial y} \tag{5.30}$$

its gradient transforms as:

$$(\partial_{x'} f',\, \partial_{y'} f') = (\partial_x f,\, \partial_y f) \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = (\nabla f)\mathbf{A^t} \tag{5.31}$$

and we can write by just adding a second line with a function g (say salinity)

$$\begin{pmatrix} \partial_{x'} f' & \partial_{y'} f' \\ \partial_{x'} g' & \partial_{y'} g' \end{pmatrix} = \begin{pmatrix} \partial_x f & \partial_y f \\ \partial_x g & \partial_y g \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = (\nabla \begin{pmatrix} f \\ g \end{pmatrix})\mathbf{A^t} \tag{5.32}$$

Now u, v are not scalar functions as are g, f but components of a vector so we have to rotate them from the prime system. And we obtain for the transformation of the strain tensor

$$\begin{pmatrix} \partial_{x'} u' & \partial_{y'} u' \\ \partial_{x'} v' & \partial_{y'} v' \end{pmatrix} = \mathbf{A} \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \mathbf{A^t} =$$
$$\begin{pmatrix} c^2\partial_x u + s^2\partial_y v + (\partial_y u + \partial_x v)sc & c^2\partial_y u - s^2\partial_x v + (\partial_y v - \partial_x u)sc \\ c^2\partial_x v - s^2\partial_y u + (\partial_y v - \partial_x u)sc & c^2\partial_y v + s^2\partial_x u - (\partial_y u + \partial_x v)sc \end{pmatrix}, \tag{5.33}$$

where $s = \sin\alpha$ and $c = \cos\alpha$. The well defined quantities that depend linearly on the strain tensor are the trace of the tensor $d = \partial_x u + \partial_y v$ and the skew-symmetric part of the tensor, which gives the vorticity $\zeta = \partial_x v - \partial_y u$. This can be easily verified looking at eq. (5.33). A well defined quadratic quantity is the determinant of the tensor $D = \partial_x u \partial_y v - \partial_y u \partial_x v$. Other well defined quadratic quantities are the square of vorticity which is called enstrophy, the square of all components of the strain matrices $H = d^2 + \zeta^2 - 2D$, the strain rate $s^2 = d^2 + \zeta^2 - 4D = H - 2D$, the Okubo-Weiss parameter $OW = s^2 - \zeta^2 = d^2 - 4D$. They are all dependent on the trace, the vorticity and the determinant. You can construct your own well-defined quantity and become famous!

If a variable does not depend on time it is called stationary. The trajectory of a small particle transported by a fluid is always tangent to the velocity vector and its speed is given by the magnitude of the velocity vector. In a stationary flow its path is called a stream line. If the flow has a vanishing divergence it can be described by a stream function Ψ with $v = \partial_x \Psi$ and $u = -\partial_y \Psi$. If the flow has a vanishing vorticity it can be described by a potential Θ with $u = \partial_x \Theta$ and $v = \partial_y \Theta$. Any vector field in 2D can be written as $u = \partial_x \Theta - \partial_y \Psi, v = \partial_y \Theta + \partial_x \Psi$, this is called the Helmholtz decomposition.

Exercise 14: Show that every flow that is described by a stream function has zero divergence.

Exercise 15: Show that every flow that is described by a potential has zero vorticity.

Exercise 16: Express the vorticity in terms of the stream function.

Exercise 17: Express the divergence in terms of the potential.

Exercise 18: Which velocity fields have zero divergence and zero vorticity?

Exercise 19: draw the velocity vectors and streamlines and calculate vorticity and divergence. Draw the a stream function where ever possible:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix};\begin{pmatrix} y \\ 0 \end{pmatrix};\begin{pmatrix} -y \\ x \end{pmatrix};\begin{pmatrix} -x \\ y \end{pmatrix};\frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix};\begin{pmatrix} \cos y \\ \sin x \end{pmatrix}. \tag{5.34}$$

5.4.2 Three Dimensional Flow

Conceptually there is not much different when going from two to three demensions. divergence is: $d = \nabla \cdot \mathbf{u} = \partial_x u + \partial_y v + \partial_z w$. vorticity is no longer a scalar but becomes a vector:

$$\zeta = \nabla \times \mathbf{u} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} = \begin{pmatrix} \partial_y w - \partial_z v \\ \partial_z u - \partial_x w \\ \partial_x v - \partial_y u \end{pmatrix}. \tag{5.35}$$

5.5 Rotation

When considering the motion of the ocean, at time scales larger than a day, the rotation of the earth is of paramount importance. Newton's laws of motion only apply when measurements are done with respect to an inertial frame, that is a frame without acceleration and thus without rotation. Adding to all measurements (and to boundary conditions) the rotation of the earth would be very involved (the tangential speed is around 400m/s and the speed of the ocean typically around 0.1m/s), one should then also have "rotating boundaries", that is the rotation would only explicitly appear in the boundary conditions, which then would be very involved. It is thus a necessity to derive Newton's laws of motion for a frame rotating with the earth, called *geocentric frame*, to make the problem of geophysical fluid dynamics treatable by calculation.

5.6 The Coriolis Force

Let us start with considering a movement of a point P that is observed by two observers, one in an inertial frame (subscript $._f$) and one in a frame (subscript $._r$) that is rotating with angular velocity Ω. The coordinates at every time t transform following:

$$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} x_r \cos(\Omega t) - y_r \sin(\Omega t) \\ x_r \sin(\Omega t) + y_r \cos(\Omega t) \end{pmatrix} \tag{5.36}$$

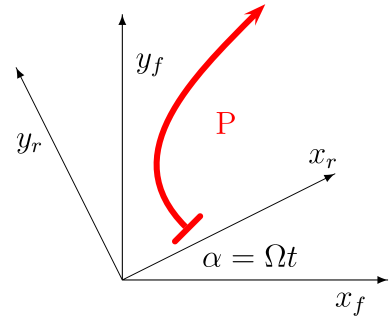


Figure 5.3: A moving point P observed by a fix and a rotating coordinate system

In a inertial (non-rotating) frame Newtons laws of motion are given by:

$$\partial_{tt}\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} F_f^x \\ F_f^y \end{pmatrix} \tag{5.37}$$

Where $F_\cdot^\cdot$ are forces per mass, to simplify notation. So, in an inertial frame if the forces vanish the acceleration vanishes too. How can we describe such kind of motion in a rotating frame.

Combining eqs. (5.36) and (5.37), performing the derivations and supposing that the forces in eq. (5.37) vanish, we obtain:

$$\partial_{tt}\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} (\partial_{tt}x_r - 2\Omega\partial_t y_r - \Omega^2 x_r)\cos(\Omega t) - (\partial_{tt}y_r + 2\Omega\partial_t x_r - \Omega^2 y_r)\sin(\Omega t) \\ (\partial_{tt}x_r - 2\Omega\partial_t y_r - \Omega^2 x_r)\sin(\Omega t) + (\partial_{tt}y_r + 2\Omega\partial_t x_r - \Omega^2 y_r)\cos(\Omega t) \end{pmatrix} = 0 \tag{5.38}$$

This is only satisfied if:

$$\begin{aligned} \partial_{tt}x_r - 2\Omega\partial_t y_r - \Omega^2 x_r &= 0 \quad \text{and} \qquad (5.39) \\ \partial_{tt}y_r + 2\Omega\partial_t x_r - \Omega^2 y_r &= 0. \qquad (5.40) \end{aligned}$$

Which is the analog of eq. (5.37) in a rotating frame.

Exercise 20: Show that eq. (5.38) is only satisfied if eqs. (5.39) and (5.40) hold.

The second and the third term in eqs. (5.39) and (5.40) look like (real) forces, especially if we write them on the right side of the equal sign and they are called the Coriolis and the centrifugal force, respectively. They also feel like real forces, when you experience them in a merry-go-round. They look like and feel like but they are no real forces. They are artifacts of a rotating coordinate system and are thus called apparent forces.

If we express this equation in terms of $u = \partial_t x$ and $v = \partial_t y$ we obtain:

$$\partial_t\begin{pmatrix} u_f \\ v_f \end{pmatrix} = \partial_t\begin{pmatrix} u_r \\ v_r \end{pmatrix} + 2\Omega\begin{pmatrix} -v_r \\ u_r \end{pmatrix} - \Omega^2\begin{pmatrix} x_r \\ y_r \end{pmatrix}. \tag{5.41}$$

But what about the real forces (F_f^x, F_f^y) we neglected? Well, forces are usually measured in the geocentric frame and so we do not have to worry how they transform from an inertial frame to a geocentric frame.

Other ways of deriving these equations can be found in literature, all leading to the same result. The equations are usually given in vector notation:

$$\partial_t \mathbf{u_f} = \partial_t \mathbf{u_r} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}). \tag{5.42}$$

Here $\times$ denotes the vector product (if you know what the vector product is: be happy!; if you do not know what the vector product is: don't worry be happy!). On our planet the rotation vector points northward along the south-north axis and has a magnitude of $|\mathbf{\Omega}| = 2\pi/T = 7.3\cdot 10^{-5}\mathrm{s}^{-1}$ Where $T \approx 24 \cdot 60 \cdot 60s$ is the earth's rotation period.

For large scale oceanic motion the horizontal component of the rotation vector $\mathbf{\Omega}$ is usually neglected, this is called the *traditional approximation*. Twice the vertical component of the rotation vector is denoted by $f = 2|\mathbf{\Omega}|\sin\theta$ and called *Coriolis parameter*, here θ is latitude. In the calculations involving mid-latitude dynamics $f = 10^{-4}\mathrm{s}^{-1}$ is a typical value.

Using the traditional approximation and restraining to the two dimensional case equation (5.42) reads:

$$\partial_t \begin{pmatrix} u_f \\ v_f \end{pmatrix} = \partial_t \begin{pmatrix} u_r \\ v_r \end{pmatrix} + f \begin{pmatrix} -v_r \\ u_r \end{pmatrix} - \frac{f^2}{4} \begin{pmatrix} x_r \\ y_r \end{pmatrix}. \tag{5.43}$$

Exercise 21: suppose $\partial_t(u_f, v_f) = (0,0)$ (no forces acting) and $(x_r, y_r) = (R\cos(\omega t), R\sin(\omega t))$ calculate ω and give an interpretation of the solution.

From now on we will omit the subscript "$._r$".

The most disturbing term on the right-hand-side of equation (5.41) is the last (centrifugal term) as it makes reference to the actual location of the particle (or fluid element) considered. This means that the laws of motion change in (rotating) space!?!

When considering the motion of a fluid we can however forget about the centrifugal term, why? For this look at figure (5.4), which shows a cylindrical tank in rotation with a fluid inside, that is rotating with the tank. What we see is, that the free surface of this fluid has a parabolic shape, which is exactly such that the pressure gradient, induced by the slope of the free fluid surface balances the centrifugal force. If this were not be the case the fluid would not be at rest! If in our calculations we suppose that the zero potential is the parabolic surface rather than a flat horizontal surface the last term in equation (5.41) is perfectly balanced by the pressure gradient due to the slope of the free fluid surface, that is:

$$g\nabla\eta + \frac{f^2}{4}\mathbf{r} = 0. \tag{5.44}$$

In such situation the last term in equation (5.41) has to be dropped.

Exercise 22: suppose $(x, y) = (R\cos(\omega t), R\sin(\omega t))$ for a fluid particle in the fluid corresponding to fig. 5.4, without exterior forces acting. Calculate ω. Such kind of motion, that is, anti-cyclonic rotation with a period which is **half** the local rotation period, is indeed often observed in oceanic and atmospheric motion and is called *inertial oscillation* and their frequency (f) is called *inertial frequency*.

On earth the same thing happens, the centrifugal force changes the geopotential of the earth, flattens it a little bit, makes it an ellipsoid. Indeed, the point on earth which has the larges distance from the centre of the earth is the summit of the Chimborazo and not the Mount Everest, but the hight of a mountain is determined with respect to the sea level.

5.7 The Shallow Water Equations in a Rotating Frame

If we take the results from the previous section we see that we only have to add the Coriolis force in the shallow water equations to obtain the shallow water equations in a rotating frame:

$$\partial_t u + u\partial_x u + v\partial_y u - fv + g\partial_x\eta = \nu\nabla^2 u \tag{5.45}$$

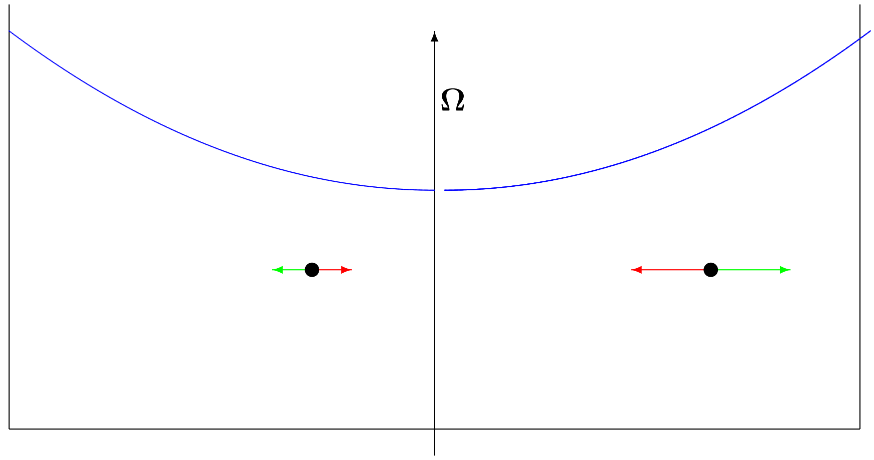


Figure 5.4: Cylinder in rotation with a free surface; two fluid particles with centrifugal force (green) and pressure gradient force (red).

$$\partial_t v + u\partial_x v + v\partial_y v + fu + g\partial_y \eta = \nu\nabla^2 v \quad (5.46)$$
$$\partial_t \eta + \partial_x \left[(H+\eta)u\right] + \partial_y \left[(H+\eta)v\right] = 0 \quad (5.47)$$
$$+\text{boundary conditions.}$$

The nonlinear terms can be neglected if *Rossby number* $\epsilon = u/(fL)$ is small. The Rossby number compares the distance a fluid particle has traveled in the time f^{-1} to the length scale of the phenomenon considered. The linear (small Rossby number) version of the shallow water equations in a rotating frame is:

$$\partial_t u - fv + g\partial_x \eta = 0 \quad (5.48)$$
$$\partial_t v + fu + g\partial_y \eta = 0 \quad (5.49)$$
$$\partial_t \eta + H(\partial_x u + \partial_y v) = 0 \quad (5.50)$$
$$+\text{boundary conditions.}$$

Important: When approaching the equator f tends to zero, so rotation no longer dominates and most of the considerations following are not applicable. Equatorial dynamics is different!

In equations (5.48) and (5.49) we have neglected the viscous term which can be safely done as ν(sea water)$\approx 10^{-6}$m^2/s.

Exercise 23: What is the Rossby number of the basin wide circulation in the North Atlantic when $u = 10^{-1}$m/s? What is the Rossby number of a Gulf Stream eddy when $u = 1$m/s and the radius $R = 30$km ?

5.8 Geostrophic Equilibrium

Large-scale ocean currents usually change on a time scale much larger than f^{-1} and are thus often well approximated by the stationary versions of eqs. (5.48) – (5.50) which are,

$$fv = g\partial_x \eta \quad (5.51)$$

$$-fu = g\partial_y \eta \tag{5.52}$$

this is called the *geostrophic equilibrium*. For a flow in geostrophic equilibrium all variables can be expressed in terms of the free surface, you can easily calculate that the vorticity is given by $\zeta = (g/f)\nabla^2\eta$. Note that to every function $\eta(x, y)$ there is a unique flow in geostrophic equilibrium associated to it. In the case with no rotation ($f = 0$) the stationary solutions of the linearised equations have $\eta = 0$ and $\partial_x u + \partial_y v = 0$. In the case without rotation $\eta(x, y)$ does not determine the flow.

Exercise 24: What happened to the stationary version of equation (5.50) ?

Exercise 25: Across the Gulf Stream, which is about 100km wide there is a height difference of approx. 1m. What is the corresponding geostrophic speed of the Gulf Stream.

Exercise 26: In a *sea surface height* (SSH) map, how can you distinguish cyclones from anti-cyclones? What happens on the southern hemisphere?

The function $\Psi = (gH/f)\eta$ is called the geostrophic *transport stream-function* as $\partial_x\Psi = Hv$ and $\partial_y\Psi = -Hu$ which means that: (i) isolines of Ψ, and also of η, are stream-lines of the geostrophic velocity field, and (ii) $\Psi(B) - \Psi(A)$ is the transport that passes between points A and B. In oceanography transport is usually measured in *Sverdrup* (1Sv $=10^6$m^3/s), which corresponds to a cube of water of side length 100 meters passing in 1 second.

When the stationarity assumption is not made the eqs. (5.48) – (5.50) can be used to derive an equation for η only. The velocity field u and v can be derived from η. This leads to:

$$\partial_t \left[\partial_{tt}\eta + f^2\eta - gH\nabla^2\eta\right] = 0 \tag{5.53}$$

$$\partial_{tt}u + f^2 u = -g(\partial_{tx}\eta + f\partial_y\eta) \tag{5.54}$$

$$\partial_{tt}v + f^2 v = -g(\partial_{ty}\eta - f\partial_x\eta) \tag{5.55}$$

Exercise 27: derive eqs. (5.53) – (5.55).

Exercise 28: show that the geostrophic equilibrium is a solution of eqs. (5.53) – (5.55).

Exercise 29: show that the only stationary solution of eqs. (5.53) – (5.55) is geostrophic equilibrium.

5.9 Energetics of flow in Geostrophic Equilibrium

For the shallow water dynamics the total energy is composed of kinetic and *available potential energy* (the part of the potential energy which is available in the layered model by reducing the surface anomaly η, if $\eta = 0$ everywhere the available potential energy vanishes):

$$E_{total} = E_{kin} + E_{pot} = \frac{\rho}{2}\int_A H(u^2 + v^2)dxdy + \frac{g\rho}{2}\int_A \eta^2 dxdy \tag{5.56}$$

$$= \frac{g^2\rho}{2f^2}\int_A H\left((\partial_x\eta)^2 + (\partial_y\eta)^2\right)dxdy + \frac{g\rho}{2}\int_A \eta^2 dxdy \tag{5.57}$$

where we used (eqs. 5.51 and 5.52). If the surface perturbation has the simple form $\eta = \eta_0\sin(x/L)$ then the energy is given by:

$$E_{total} = E_{kin} + E_{pot} = \frac{g\rho\eta_0^2}{4}\int_A \left(\frac{Hg}{f^2L^2} + 1\right)dxdy \tag{5.58}$$

Were the first term is the kinetic and the second term the available potential energy. We see that in a geostrophic flow the kinetic energy is larger than the available potential energy when the structure is smaller than the *Rossby radius* $R = \sqrt{gH/f^2}$. So for large geostrophic structures most of the energy is in the potential part and for small structures in the kinetic part. The *Rossby radius* is of the order of a few thousands of kilometers for the shallow water dynamics of the ocean (the barotropic Rossby radius) but only several tenths of kilometers when the reduced gravity dynamics of the layer above the thermocline are considered (the baroclinic Rossby radius).

5.10 The Taylor-Proudman-Poincaré Theorem

The geostrophic solution can also be dervied from the rotating Navier-Stokes equations. When time dependence, viscous effects, the horizotal component of the rotation vector and the non-linear terms are neglected, they simplify to:

$$-fv + \frac{1}{\rho_0}\partial_x P = 0 \quad (5.59)$$

$$fu + \frac{1}{\rho_0}\partial_y P = 0 \quad (5.60)$$

$$\partial_z P = -g\rho \quad (5.61)$$

$$\partial_x u + \partial_y v + \partial_z w = 0 \quad (5.62)$$

When horizontal changes of density vanish, we obtain using eq. (5.61): $\partial_{xz}P = \partial_{yz}P = 0$ which leads, using eqs. (5.59), (5.60), to

$$\partial_z u = \partial_z v = 0. \quad (5.63)$$

Furthermore, deriving (5.59) by y and (5.60) by x and taking the difference gives $\partial_x u + \partial_y v = 0$, together with (5.62) we get:

$$\partial_z w = 0 \quad (5.64)$$

The non variation of the velocity field with the vertical direction is usually refered to as the Taylor-Proudman-Poincaré theorem. It now justifies the neglecting of the horizontal shear in the derivation of the rotating shallow water equations above.

Please note, that based on the shallow water equations we derived in section 5.1, that if there is no vertical shear this property is conserved. Now, in the dynamics dominated by rotation the statement is much stronger, as it says that there is no variation of the three components of the velocity field in the vertical direction.

5.11 Linear Potential Vorticity and the Rossby Adjustment Problem

If we take ∂_x (eq. (5.49)) - ∂_y (eq. (5.48)) we see that:

$$\partial_t \zeta + f(\partial_x u + \partial_y v) = 0. \quad (5.65)$$

relating vorticity $\zeta = \partial_x v - \partial_y u$ to divergence $\partial_x u + \partial_y v$. Using eq. (5.50) we get:

$$\partial_t \left(\frac{\zeta}{f} - \frac{\eta}{H} \right) = 0. \quad (5.66)$$

One usually calls $Q_{sw}^{lin} = \frac{\zeta}{H} - \frac{f\eta}{H^2}$ the linear shallow water potential vorticity. The above equations show, that at every location *linear shallow water potential vorticity* (PV) is conserved, when the dynamics is governed by the linearised shallow water equations.

The Rossby adjustment problem considers the adjustment of an initially step-like perturbation (see fig. 5.11), and we would like to know the final, geostrophically balanced state of this perturbation. To this end we use the conservation of potential vorticity and we further require the final state to be in geostrophic equilibrium. The initial potential vorticity is given

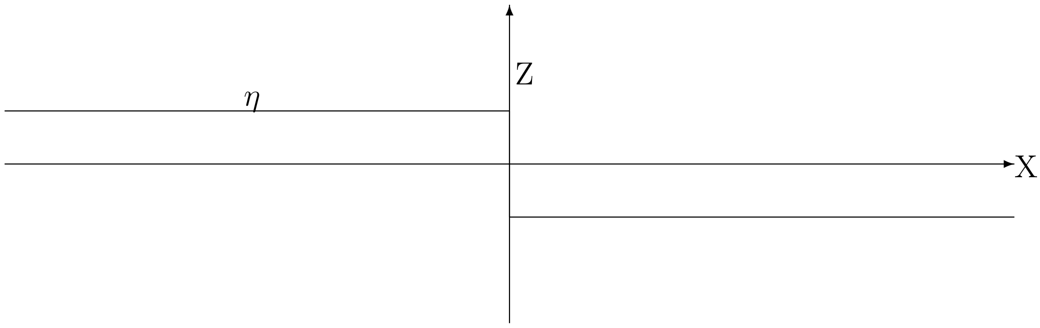


Figure 5.5: Initial condition

by $sgn(x)(f\eta_0)/H^2$ the PV of the adjusted state is the same, we thus have,

$$g/(Hf)\partial_{xx}\eta_a - f\eta_a/H^2 = sgn(x)(f\eta_0)/H^2, \tag{5.67}$$

$$R^2\partial_{xx}\eta_a - \eta_a = \eta_0 sgn(x), \tag{5.68}$$

which has the solution:

$$\eta_a = sgn(x)\eta_0(exp(-|x|/R) - 1) \tag{5.69}$$

with $R = \sqrt{gH/f^2}$ is called the *Rossby radius* of deformation. It is the distance, a gravity wave travels in the time f^{-1}.

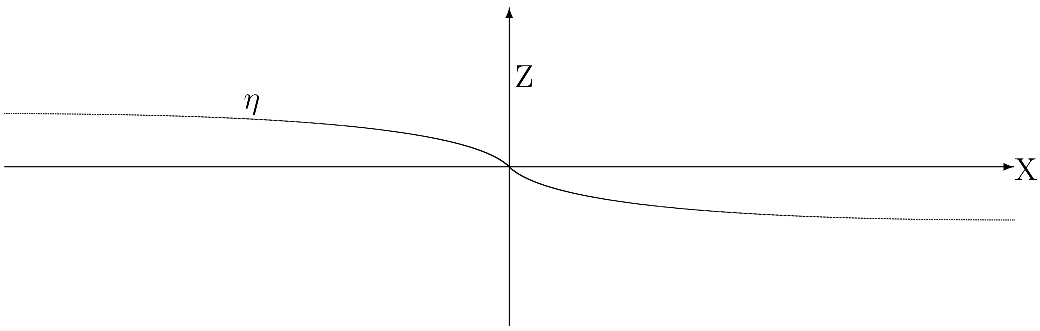


Figure 5.6: Adjusted state

We have calculated the final geostrophically adjusted state from an initial perturbation using geostrophy and conservation of linear PV, but we have not shown how this adjustment happens. For this a numerical integration of the linear shallow water equations are necessary, eqs. (5.48) – (5.50).

Exercise 30: Calculate the final velocity field (u, v).

Exercise 31: What happens when rotation vanishes?

Exercise 32: In section 5.2 we saw that if rotation is vanishing, an initial perturbation of the free surface moves away in both directions leaving an unperturbed free surface and a zero velocity behind. Does this contradict the conservation of linear potential vorticity?

Exercise 33: Calculate the loss of available potential energy and compare it to the gain in kinetic energy during the adjustment process.

Exercise 34: Calculate the (barotropic) Rossby radius of deformation ($H = 5$km), calculate the reduced gravity (baroclinic) Rossby radius of deformation ($H = .5$km, $g' = 3. \cdot 10^{-2}$m/s^{-2})

Exercise 35: * Inertia-gravity waves progressing in th y-direction are given by:

$$\begin{aligned}
\eta(y,t) &= \eta_0 cos(k_y y - \omega t) \\
u(y,t) &= \eta_0 \frac{-g k_y f}{f^2 - \omega^2} sin(k_y y - \omega t) \\
v(y,t) &= \eta_0 \frac{-g k_y \omega}{f^2 - \omega^2} cos(k_y y - \omega t),
\end{aligned}$$

the dispersion relation is: $f^2 - \omega^2 + gHk_y^2 = 0$. Show that inertia-gravity waves have vanishing potential vorticity.

5.12 Potential Vorticity (non-linear)

Similar calculations for the non-linear equations (5.45) – (5.47) lead to

$$\frac{d}{dt}\left(\frac{\zeta + f}{H + \eta}\right) = 0. \tag{5.70}$$

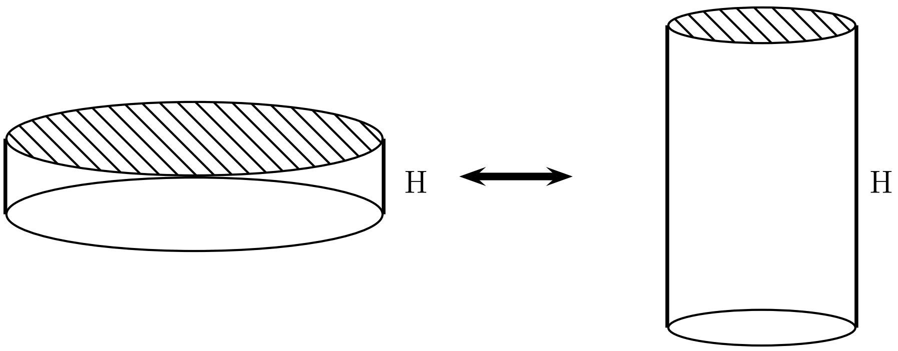
This means that every fluid parcel, or in this case every fluid column, conserves its potential vorticity $Q_{sw} = (\zeta + f)/(H + \eta)$, that is, potential vorticity is transported by the two dimensional flow. The part $f/(H + \eta)$ which does not depend explicitly on the velocity is called *planetary potential vorticity*, while $\zeta/(H + \eta)$ is called the dynamical part.

Example: Eddy over sea mount.

Exercise 36: derive eq. (5.70).

Exercise 37: If you make the assumption of linearity, can you obtain Q_{sw}^{lin} from Q_{sw}?

Exercise 38: The moment of inertia of a cylinder of mass m, radius r and height H is given by $I = mr^2/2$ the angular momentum is given by $L = I\omega$. If a cylinder stretches or flatness without any forces acting from the outside its angular momentum is conserved:



Show, that during this process ω/H is conserved.

The previous exercise demonstrates, that the conservation of potential vorticity is nothing else than the conservation of angular momentum applied to a continuum in a rotating frame.

5.13 The Beta-plane

So far we supposed the earth to be flat! The dominant difference, induced by the spherical shape of the earth, for the large scale ocean dynamics, at mid- and low latitudes, is the change of the (locally) vertical component of the rotation vector.

For the large scale circulation a major source of departure from geostrophy is the variation of $f = f_0$ with latitude. So far we have considered f to be constant we will now approximate it by $f = f_0 + \beta y$, where $f_0 = 2|\mathbf{\Omega}|\sin(\theta_0)$ and $\beta = 2(|\mathbf{\Omega}|/R)\cos(\theta_0)$ are constant where R is the radius of the earth, it takes its maximum value $\beta_{max} = 2.3 \times 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}$ at the equator. The geometry with a linearly changing Coriolis parameter is called the *β-plane*. The change of f with latitude, the so called *β-effect*, compares to the effect of constant rotation for phenomena with horizontal extension $L \approx f/\beta = R\tan(\theta)$ or larger.

Replacing f by $f_0 + \beta y$ in equations (5.48), (5.49) and (5.50):

$$\partial_t(\partial_x v - \partial_y u) + f(\partial_x u + \partial_y v) + \beta v = 0 \tag{5.71}$$

leading to:

$$\partial_t \zeta - f\partial_z w + \beta v = 0. \tag{5.72}$$

which states, that the vorticity ζ is changed by the vertical gradient of the vertical velocity (vortex stretching) and the planetary vorticity change, due to β and the latitudinal velocity).

Exercise 39: what is the sign of f and β on the northern and southern hemisphere, respectively?

Exercise 40: what is the value of f and β on the equator, north and south pole?

Exercise 41: discuss the importance of f and β for equatorial dynamics.

5.14 A few Words About Waves

As mentioned in the preface we do not explicitly consider wave dynamics in this introductory text. I like to make, nevertheless, some “hand waving” arguments about the role of waves in the ocean.

The ocean and atmosphere dynamics at large scales are always close to a geostrophic balance. There are, however, different sources of perturbations of the geostrophically balanced state:

- variation of the Coriolis parameter f
- non-linearity
- topography
- instability
- forcing (boundary conditions)
- friction
- other physical processes (convection, ..)

As the geostrophic adjustment process happens on a much faster time scale than the geostrophic dynamics, these perturbations lead not so much to a departure from the geostrophic state but more to its slow evolution. In this adjustment process, discussed in section 5.11, (gravity) waves play an important role. It is an important part of research in geophysical fluid dynamics *(GFD)* (DFG, en français) to find equations that reflect the slow evolution of the geostrophic state, without explicitly resolving the geostrophic adjustment process. Such equations are called *balanced equations*, and are based on the evolution of PV. The best known system of balanced equations are the *quasi-geostrophic equations*. The problem in constructing such equations is how to calculate the velocity field from PV, a process usually referred to as inversion. The fast surface gravity waves influenced by rotation, *Poincaré waves* have no PV signature and thus do not appear in the balanced equations, which leads to a large simplification for analytical and numerical calculations. Balanced equations such rely on the assumption that the ocean dynamics can be separated into fast wave motion and slow vortical motion with no or negligible interactions between the two. They describe the dynamics on time-scales longer than the period of gravity waves, typically several inertial periods f^{-1}. The balanced equations are not valid when approaching the equator, as $f^{-1} \to \infty$. The dynamics described by the balanced equations is said to represent the slow dynamics or to evolve on the *slow manifold* .

Balanced equations explicitly resolve the *Rossby waves* which play a key role in the response of balanced dynamics to forcing and the adjustment to a geostrophic state.

The very fast dynamics is the dynamics that happens at a time scale smaller than f^{-1} and it is usually three dimensional turbulent dynamics. To model it the full three dimensional Navier-Stokes equations have to be considered.

Chapter 6

Gyre Circulation

The ocean is forced at its surface by a wind-stress $\tilde{\tau}$ which is measured in Newton/m^2. A typical value for the ocean is in the order of 0.1N/m^2. In the present manuscript we work with $\tau = \tilde{\tau}/\rho$ which has units of m^2/s^2.

6.1 Sverdrup Dynamics in the SW Model (the math)

In all the ocean basins an almost stationary large scale *gyre* circulation is observed. We suppose that this circulation is a consequence of the wind shear at the ocean surface. We thus add some (wind) forcing to the linearized stationary shallow water equations on the β-plane.

$$
\begin{aligned}
-fv + g\partial_x\eta &= \tau_x/H && (6.1)\\
+fu + g\partial_y\eta &= \tau_y/H && (6.2)\\
H(\partial_x u + \partial_y v) &= 0 && (6.3)\\
&\text{+boundary conditions.}
\end{aligned}
$$

Adding $-\partial_y$ (6.1) and ∂_x (6.2) leads to:

$$
H\beta v = (\partial_x\tau_y - \partial_y\tau_x) \tag{6.4}
$$

So at every point the meridional component of the fluid transport (vH) is completely determined by the vorticity of the surface stress! Equation (6.4) is called the *Sverdrup relation.* It says that if vorticity is injected into the fluid parcel it can not increase its vorticity as this would contradict stationarity, so it moves northward where *planetary potential vorticity* (f/H) is larger. So the Sverdrup relation is a statement of conservation of potential vorticity in a forced and stationary situation.

When knowing the wind field, the Sverdrup relation gives v, using the zero divergence of geostrophic flow we can calculate $\partial_x u$. If we know u at one point in a ocean basin at every latitude we can determine u in the whole basin by integrating in the zonal direction,

$$
u(x1, y1) = -\int_{x0}^{x1} \partial_y v(x, y1)dx + u(x0, y1). \tag{6.5}
$$

But u is prescribed at the two boundaries of the ocean basin (as the velocity vector at the boundary is directed parallel to the boundary), which makes u over-determined. What does this mean in "physical terms?" Take a look at fig. 6.1, where a caricature of the North Atlantic with a simplified wind-stress (independent of longitude) is given. The corresponding v component of the velocity is also given. If we start by imposing a vanishing zonal velocity at the eastern boundary the stream lines will look as in fig. 6.1 (if we impose it at the western boundary the picture will flip with respect to a vertical line). It is clearly seen that stream lines intersect the western boundary, which means, that there is flow through the western boundary. This is contrary to the concept of a boundary.

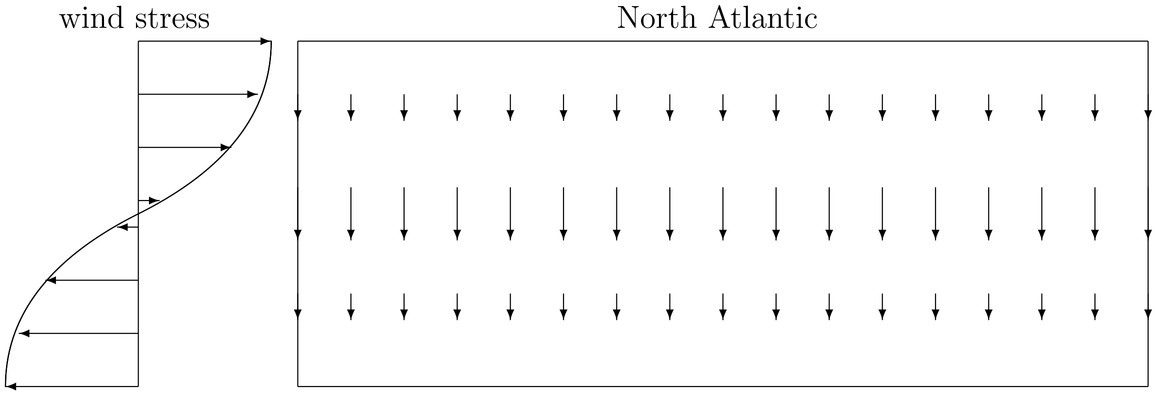


Figure 6.1: Sverdrup balance: v-component only

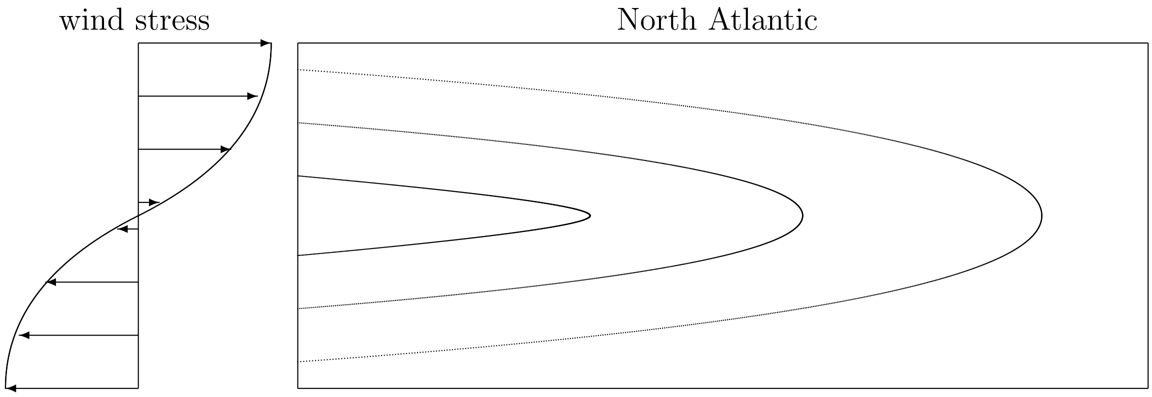


Figure 6.2: Sverdrup balance: Stream function with u=0 on eastern boundary

How can we solve the problem? Why not, for a change, look at the real ocean. Measurements of the ocean indicate that the circulation is, after all, well represented by (6.1) with the only difference of a strong *western boundary currents* which we do not have in (6.1). The western boundary current, which closes the Sverdrup circulation, is dominated by friction (eddy viscosity effects). From a conceptual view point it is clear that such an area is necessary and that each stream line has to pass by such an area, as the wind constantly injects (negative) vorticity (and energy) in the ocean, that has to be dissipated somewhere. But lets be more quantitative. When including friction at the western or eastern boundary we have to change eq. (6.2) to:

$$fu + g\partial_y\eta = \tau_y/H + \nu_{eddy}\partial_{xx}v. \tag{6.6}$$

The term $\nu_{eddy}\partial_{xx}v$ represents the dominant contribution of friction as it is the meridional velocity component v that changes fastest in the zonal x-direction. The eddy viscosity ν_{eddy} is many orders of magnitude larger than the molecular viscosity of sea water. The concept of eddy viscosity is explained in section 10.2. Near the boundary we can neglect the wind forcing and the dominant balance is then,

$$\beta v_B = \nu_{eddy} \partial_{xxx} v_B, \tag{6.7}$$

which has solutions of the form,

$$v_B = C_1 \exp(2x/r) + \exp(-x/r) \left(C_2 \cos(-x/\tilde{r}) + C_3 \sin(-x/\tilde{r}) \right), \tag{6.8}$$

with $r = (\nu_{eddy}/\beta)^{1/3}/2$ and $\tilde{r} = r/\sqrt{3}$. One condition of the boundary solution is, that it has to decrease away from the boundary, which means that $C_1 = 0$ if the boundary current develops on the western boundary and $C_2 = C_3 = 0$ if the boundary current develops on the eastern boundary. The boundary dynamics is there to insure that $u = v = 0$ on the boundary, these are two conditions. If the boundary current is on the eastern boundary we have only one constant to adjust, so it is usually not possible. So the frictional boundary current can do its job (satisfy the boundary conditions) only if it is on the western boundary. The solution of the boundary layer obtained is refered to as the *Munk layer*.

There are still other dynamical arguments why the boundary current can not be on the eastern boundary: (i) in the situation in fig. 6.1 the wind injects negative vorticity in the flow, vorticity is conserved by the fluid column moving with the flow, not subject to any forcing. In a stationary state the vorticity extracted has to be re-injected during the cyclical path of a fluid column. A boundary layer at the western border does exactly this. A boundary layer at the eastern border would drain even more vorticity, which leads to a contradiction in terms of the vorticity balance. (ii) The dynamical adjustment in the ocean is performed by Rossby waves, which have a westward group velocity. This means that the dynamics at a point adjusts to the dynamics to its eastern side. That's what the boundary current does, so it has to be to the extreme western part of the basin to adjust to the entire interior dynamics.

On the southern hemisphere the boundary current is also on the western boundary as β (unlike f) has the same sign on both hemispheres! In the above derivation of the Sverdrup transport only β but not f was involved.

So the big picture is: (i) the ocean interior is well described by Sverdrup dynamics, (ii) which is complemented at the western boundary by a thin boundary current, which is dominated by friction.

Comment 1: The wind stress induces a transport (uH, vH) rather than a velocity (u, v).

Exercise 42: which dynamics would we expect in fig. 6.1 when rotation vanishes?

Exercise 43: in the above calculations we have neglected the non-linear terms. This is only valid when the Rossby numbers are small. What is the Rossby number of the interior flow at mid latitudes when $v = .1$m/s, $L = 5000$km. What is the Rossby number of the boundary layer flow at mid latitudes when $v = 1.0$m/s, $L = 100$km.

Comment 2: For the Sverdrup relation to apply, it is not so much the Rossby number that has to be small but the two terms neglected, (i) the time derivative of the *relative vorticity* $\partial_t \zeta$ and (ii) the non-linear term $\mathbf{u}\nabla\zeta$, have to be small compared to the transport of *planetary vorticity* $v\beta$. Observations show that the mean wind forcing and thus the mean circulation changes only slightly during several years in large parts of the worlds ocean. The total vorticity, measured from an inertial frame, of the fluid motion on our planet can be decomposed in the relative part, measured from a frame moving (rotating) with the surface of the earth, and the planetary part given by the Coriolis parameter f. In the boundary layer, however, the non-linear term is not smaller than the transport of planetary vorticity and there are non-linear phenomena in the western boundary currents, as for example the Gulf-Stream and the Kuroshio, which are not well explained by the above theory.

6.2 The Ekman Layer

Strictly speaking, this section belongs to chapter 10 because it deals with how wind forcing penetrates to the deep ocean, but it just happens that we need to know Ekman theory to continue our investigation of the gyre circulation.

Ekman's theory of the adjustment of a fluid in a rotating frame to an equilibrium when subject to wind forcing, is probably the most cited and most misunderstood theory of ocean dynamics. To elucidate this Ekman layer dynamics we will advance in small steps, emphasizing the physical understanding of the process, without neglecting the mathematical derivation.

Suppose we have an infinitely deep layer of a homogeneous fluid subject to wind forcing τ_x, constant in time and space, at its surface that is acting in the x-direction. The flow is independent of x, y as the forcing has no variations in these variables and as there are no boundaries. But the flow depends on the vertical coordinate z. In this case the vertical velocity w vanishes everywhere due to the divergence free condition, eq. (5.4). The Navier-Stokes equations (5.1) – (5.4), in a rotating frame, then simplify to:

$$\partial_t u(z,t) - fv(z,t) = \nu \partial_{zz} u(z,t) \tag{6.9}$$
$$\partial_t v(z,t) + fu(z,t) = \nu \partial_{zz} v(z,t) \tag{6.10}$$

with the boundary conditions:

$$\nu \partial_z u(0) = \tau_x; \;\; \partial_z v(0) = 0, \tag{6.11}$$
$$\lim_{z \to -\infty} \partial_z u(z) = \lim_{z \to -\infty} \partial_z v(z) = 0. \tag{6.12}$$

The surface boundary condition (6.11) represents the vertical gradient of the horizontal velocity due to wind stress, while we suppose no frictional forces at the (far away) bottom of the Ekman layer.

6.2.1 Ekman Transport (one layer)

To further simplify the problem we consider the transport $U(t) = \int_{-H}^{0} u(z,t)dz$ and $V(t) = \int_{-H}^{0} v(z,t)dz$ of the whole fluid column. Please note, that these variables depend only on time and we have neglected all vertical structure in the problem. This can be easily done in the present problem as eqs. (6.9), (6.10) and the boundary conditions (6.11), (6.12) are linear. Integrating the right hand side of eq. (6.9) we have $\int_{-H}^{0} \nu \partial_{zz} u(z,t)dz = \nu \partial_z u(0) = \tau_x$. When we further neglect friction at the bottom of the fluid layer eq. (6.9), (6.10) and boundary conditions (6.12) read:

$$\partial_t U(t) - fV(t) = \tau_x \tag{6.13}$$
$$\partial_t V(t) + fU(t) = 0 \tag{6.14}$$

We now like to consider the spin up of an Ekman transport initially at rest. In the non-rotating case ($f = 0$) we have the solution:

$$U(t) = \tau_x t \tag{6.15}$$
$$V(t) = 0, \tag{6.16}$$

so the fluid constantly accelerates in the x-direction and no stationary state is reached!

In the rotating case ($f \neq 0$) the solution is given by:

$$U(t) = \frac{\tau_x}{f} \sin(ft) \quad (6.17)$$

$$V(t) = \frac{\tau_x}{f} (\cos(ft) - 1) \quad (6.18)$$

Initially the solution behaves as in the non rotating case, that is, it accelerates in the x-direction with an acceleration given by τ_x. But in the rotating case eqs. (6.13) and (6.14) also have the stationary (time-independent) solution:

$$U = 0; \quad V = -\frac{\tau_x}{f}, \quad (6.19)$$

which has no counter part in the non-rotating case. This solution is a force balance between the Coriolis force and the wind stress. The depth averaged Ekman transport is at 90^o to the right of the wind force as this is the only possibility for the Coriolis force to balance the wind stress.

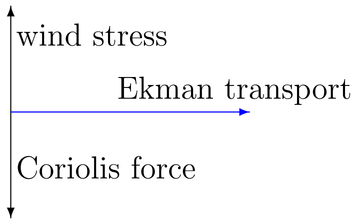


Figure 6.3: Depth averaged Ekman transport

The solutions for the rotating case given in eqs. 6.17 and 6.18 are in fact a sum of the stationary solution plus *inertial oscillation*. When friction is included the oscillations will be damped and the transport will converge towards a (modified) stationary solution.

Exercise 44: What is the energetics of the Ekman transport?

Exercise 45: Does the Ekman transport depend on the viscosity?

It is the Ekman transport, and only the Ekman transport, that determines the influence of the wind forcing on the deep ocean. For completeness we will discuss in the next subsection the vertical structure of the Ekman dynamics.

6.2.2 The Ekman Spiral

We start this section with two instructive exercises.

Exercise 46: What happens when we include bottom (Rayleigh) friction in eqs. 6.13 and 6.14? The stationary solution is governed by:

$$-fV = -rU + \tau_x \quad (6.20)$$

$$fU = -rV \quad (6.21)$$

and we obtain the solution:

$$U = \frac{r}{r^2 + f^2}\tau_x \quad (6.22)$$

$$V = \frac{-f}{r^2 + f^2}\tau_x. \quad (6.23)$$

We see that when including bottom friction the fluid motion is still deviated to the right (on the northern hemisphere) with respect to the wind stress but the angle is smaller than the 90^o of the frictionless case. So the friction induces fluid motion in the direction of the wind stress.

Exercise 47: Two layers: We now suppose that the Ekman layer can be decomposed into two layers of thickness H_1 and H_2. This "poor man's vertical structure" does not correspond to any real situation but helps us to understand the physics of the Ekman spiral treated in the next subsection. The governing equations for the stationary solution are:

$$\begin{aligned} -fV_1 &= r(U_2/H_2 - U_1/H_1) + \tau_x && (6.24)\\ fU_1 &= r(V_2/H_2 - V_1/H_1) && (6.25)\\ -fV_2 &= r(U_1/H_1 - U_2/H_2) && (6.26)\\ fU_2 &= r(V_1/H_1 - V_2/H_2) && (6.27) \end{aligned}$$

Where r times the velocity difference represents the linear friction between the two layers. You can write the linear system (6.24) – (6.27) in the form,

$$\mathbf{AU} = \mathbf{B}, \tag{6.28}$$

where $\mathbf{U} = (U_1, V_1, U_2, V_2)$. Verify that all solutions have: $U_1 + U_2 = 0$, and $f(V_1 + V_2) = -\tau_x$, which is the Ekman transport already calculated above. You can use this to eliminate U_2 and V_2 from the problem and simplify eq. (6.28) to:

$$\tilde{\mathbf{A}}\tilde{\mathbf{U}} = \tilde{\mathbf{B}} \tag{6.29}$$

with $\tilde{\mathbf{U}} = (U_1, V_1)$. Find the solution and give an interpretation.

What is the vertical structure of the Ekman transport? We can approximate the vertical structure by including more and more layers in the vertical. The first layer being subject to wind forcing, the Coriolis force and the friction induced by the second layer. Every other layer is driven by the frictional force transmitted by is upper neighbour and feels the friction of its lower neighbour. All layers are subject to the Coriolis force. Using the results from subsection 6.2.1 we estimate that every layer will move to the right of the movement of its upper neighbour, at a smaller pace. Such motion will lead to a spiral motion in the vertical decaying with depth. To render this qualitative arguments into a quantitative theory we go back to eqs. (6.9), (6.9) and boundary conditions (6.11), (6.12). To simplify the problem we will only consider the time independent solution of these equations neglecting the inertial oscillations. Equations (6.9) and (6.10) can be combined to form a single equation of fourth order:

$$-f^2 u(z) = \nu^2 \partial_{zzzz} u. \tag{6.30}$$

If we suppose that the viscosity is independent z-component this equation is easily solved and the solution satisfying the boundary conditions 6.11 6.12 is:

$$\begin{aligned} u(z) &= V_0 \exp(z/\delta)\cos(z/\delta + \pi/4) && (6.31)\\ v(z) &= V_0 \exp(z/\delta)\sin(z/\delta + \pi/4) && (6.32) \end{aligned}$$

where $\delta = \sqrt{2\nu/|f|}$ is the Ekman layer thickness and $V_0 = \tau_x \delta/(\nu\sqrt{2})$. The solution shows that the current at the surface is deviated 45^o to the right with respect to the wind velocity (on the northern hemisphere, to the left on the southern hemisphere).

Exercise 48: What is the energetic balance of the Ekman spiral?

An Ekman spiral is clearly observed in the ocean where $\delta \approx 30$m, in laboratory experiments and in numerical experiments. Indeed the work of Vagn Walfrid Ekman (1905) was initiated by Fridtjof Nansen who observed that in the Arctic the ice drifts 20^o to 40^o to the right of the wind direction and who also had the physical intuition that rotation of the earth was the reason and that the resulting dynamics should be a spiral decreasing with depth. He then encouraged Vagn Walfrid Ekman (1905) to do the mathematics.

At large Reynolds numbers the dynamics in the Ekman layer is turbulent leading to an eddy viscosity that varies with depth and the spiral is distorted. We emphasize once more, that the Ekman transport however does not depend on the internal structure and details of the Ekman layer, as demonstrated in subsection 6.2.1. It is this transport that puts the deep ocean into motion.

It is no surprise that the Ekman spiral was discovered through measurements of the drift of sea ice and the currents underneath. First, it is much easier to perform current measurements by drilling a small whole in the ice and descending the current meter, than to perform the same kind of measurements from a drifting ship in a wavy ocean. Second, the damping of surface waves in the ocean, by ice cover, reduces small-scale dynamics (turbulence) that overlay or perturb the Ekman spiral, and which distorts the Ekman spiral. The deviation of the surface current to the wind direction is indeed smaller in the ice free ocean, usually around 30^o.

We note that the Ekman transport does only depend on the shear (τ_x, τ_y) and the Coriolis parameter. The role of friction is to set the depth and the structure of the dynamics in the Ekman layer. An Ekman dynamics exists not only at the ocean surface but also at the ocean floor, that exerts a frictional force on the fluid.

The large difference between the Ekman and the geostrophic dynamics is its variation with depth. In the geostrophic dynamics the force is due to the horizontal pressure gradient, which has no variation with depth in a homogeneous ocean when the hydrostatic approximation is made. Whereas the Ekman dynamics relies on (turbulent; see Section 10.2) viscosity to penetrate the depth of the ocean. The Ekman dynamics is thus confined to the upper tenths of meters of the ocean.

6.3 Sverdrup Dynamics in the SW Model (the physics)

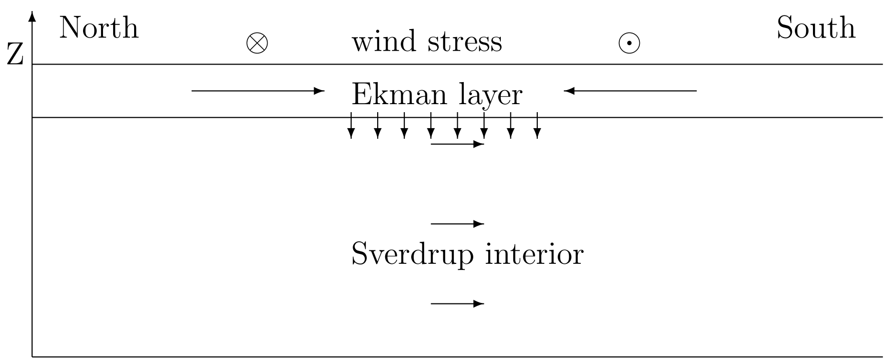


Figure 6.4: Sverdrup physics

In section 6.1 we have calculated the potential vorticity balance of the of the stationary large scale oceanic dynamics of a shallow fluid layer subject to wind forcing at the surface.

From what we learned in section 6.2 it seems, at first sight, unlikely that a fluid layer, that is forced by a wind stress at the surface will develop a velocity independent of depth. It seems much more likely that a substantial shear will develop in the upper-part (Ekman-layer) of the fluid, and that the main body of the fluid rests motionless. This is however not the case, the wind-stress is indeed transferred to the deep layers. How this happens is the subject of this section.

As we have seen in section 6.2 the transport in the Ekman layer ($H_{Ek} \approx 30$m) is given by,

$$u_{Ek} H_{Ek} = \tau_y / f \qquad (6.33)$$

$$v_{Ek} H_{Ek} = -\tau_x / f \qquad (6.34)$$

using the zero divergence integrated over the Ekman layer:

$$\int_{-H_{Ek}}^{0} \partial_x u + \partial_y v + \partial_z w \, dz = 0 \ \rightsquigarrow \ H_{Ek}(\partial_x u_{Ek} + \partial_y v_{Ek}) = -w(0) + w(-H_{Ek}) = -w_{Ek}, \qquad (6.35)$$

we see that the Ekman dynamics leads to a vertical velocity (Ekman-pumping):

$$w_{Ek} = -\partial_x(\tau_y/f) + \partial_y(\tau_x/f). \qquad (6.36)$$

In the geostrophic interior no direct action of the wind-stress is felt and eqs. (6.1) – (6.3) give,

$$\beta v = f \partial_z w, \qquad (6.37)$$

which is called the Sverdrup relation. On the surface w_{Ek} has to be compensated by a vertical "geostrophic" velocity $w_G = -w_{Ek}$. Using eq. (6.4) we get,

$$\beta H v_G = f w_G = -f w_{Ek} = f H_{Ek}(\partial_x u_{Ek} + \partial_y v_{Ek}) = f \left[\partial_x(\tau_y/f) - \partial_y(\tau_x/f) \right]. \qquad (6.38)$$

The total zonal (Sverdrup) transport is,

$$H v_S = H v_G + H_{Ek} v_{Ek} = f/\beta \left[\partial_x(\tau_y/f) - \partial_y(\tau_x/f) \right] - \tau_x/f \qquad (6.39)$$

$$= (\partial_x \tau_y - \partial_y \tau_x)/\beta \qquad (6.40)$$

which is identical to 6.4!

What do all this beautiful calculations tell us?

- The Sverdrup transport can be split up between an Ekman transport and a geostrophic interior transport.
- The Ekman transport is directly set into motion by the by the wind stress through (eddy) viscous friction.
- The interior dynamics is set up by the vertical velocity induced by the divergence of the Ekman transport
- The interior dynamics is put into motion by stretching of the water column and the conservation of planetary potential vorticity (f/H).

Chapter 7

Multi-Layer Ocean dynamics

7.1 The Multilayer Shallow Water Model

The models employed so far to study the ocean dynamics consisted of a single layer, which represented the dynamics of a single vertically homogeneous (in speed and density) layer above a solid bottom or above a infinitely deep inert layer of higher density (reduced gravity model). We also saw that these type of models are very successful in explaining the main features of the large scale ocean circulation. There are, however, important phenomena of the circulation which can not be explained by such one-layer models. We thus move on to the dynamics of several layers of homogeneous (in speed and density) fluid layers of different density and velocity, stacked one above the other. We will here restrict the analysis to a model with two active layers, the generalisation to more layers is straightforward. The equations governing the dynamics of such a hydrostatic two-layer shallow water model are:

$$\begin{aligned}
\partial_t u_1 + &\quad u_1\partial_x u_1 + v_1\partial_y u_1 - f v_1 + g\partial_x(\eta_1+\eta_2) = &\nu\nabla^2 u_1 \qquad (7.1)\\
\partial_t v_1 + &\quad u_1\partial_x v_1 + v_1\partial_y v_1 + f u_1 + g\partial_y(\eta_1+\eta_2) = &\nu\nabla^2 v_1 \qquad (7.2)\\
\partial_t \eta_1 + &\quad \partial_x\left[(H_1+\eta_1)u_1\right] + \partial_y\left[(H_1+\eta_1)v_1\right] = &0 \qquad (7.3)\\
\partial_t u_2 + &\quad u_2\partial_x u_2 + v_2\partial_y u_2 - f v_2 + g''\partial_x(\eta_1+\eta_2) + g'\partial_x\eta_2 = &\nu\nabla^2 u_2 \qquad (7.4)\\
\partial_t v_2 + &\quad u_2\partial_x v_2 + v_2\partial_y v_2 + f u_2 + g''\partial_y(\eta_1+\eta_2) + g'\partial_y\eta_2 = &\nu\nabla^2 v_2 \qquad (7.5)\\
\partial_t \eta_2 + &\quad \partial_x\left[(H_2+\eta_2)u_2\right] + \partial_y\left[(H_2+\eta_2)v_2\right] = &0 \qquad (7.6)\\
&\quad +\text{boundary conditions} &.
\end{aligned}$$

Where the index 1 and 2 denote the upper and the lower layer, respectively. It is interesting to note that the two layers interact only through the hydrostatic pressure force caused by the thicknesses of the layers. Indeed, the upper layer (layer 1) is subject to the hydrostatic pressure of the surface which has a total anomaly of $\eta_1+\eta_2$. Whereas the lower layer (layer 2) is subject to the same pressure plus the pressure at the interface $g'\eta_2$ due to the increased density in the lower layer, where $g' = g(\rho_2-\rho_1)/\rho_2$ is the reduced gravity, that is, the weight of the lower-layer fluid in the upper layer environment and $g'' = g\rho_1/\rho_2$. In the *Boussinesq approximation* g'' is set equal to g, thus neglecting the density differences in the inertial mass but keeping it in the weight. Equations (7.1) – (7.6) are the mathematical model for the investigations of the present chapter.

Exercise 49: What happens to equations (7.1) – (7.6) if $\rho_1 = \rho_2$?

Exercise 50: Write down the linearised version of eqs. (7.1) – (7.6).

7.2 Conservation of Potential Vorticity

Exercise 51: Show that the linearised version of eqs. (7.1) – (7.6) conserve the linear potential vorticity at every horizontal location and for every layer.

Exercise 52: Show that eqs. (7.1) – (7.6) conserve the potential vorticity at every fluid coloumn advected by the flow and for every layer (when friction is neglected).

Wow! This means that if we describe our ocean by more and more layers, then potential vorticity is conserved for every fluid particle!

7.3 Geostrophy in a Multi-Layer Model

As we have seen in section 5.8 the geostrophic equilibrium is a balance between the pressure and the Corilolis force, neglecting time-dependence, non-linearity, friction and using the Boussinesq approximation $g'' \approx g$, eqs. (7.1) – (7.6) then read:

$$\begin{aligned} fv_1 &= g\partial_x(\eta_1 + \eta_2) && (7.7)\\ -fu_1 &= g\partial_y(\eta_1 + \eta_2) && (7.8)\\ fv_2 &= g\partial_x(\eta_1 + \eta_2) + g'\partial_x\eta_2 && (7.9)\\ -fu_2 &= g\partial_y(\eta_1 + \eta_2) + g'\partial_y\eta_2 && (7.10) \end{aligned}$$

It is now interesting to consider the differences between eqs. (7.7) – (7.9) and (7.8) – (7.10) which are:

$$v_1 - v_2 = -\frac{g'}{f}\partial_x\eta_2 \qquad (7.11)$$

$$u_1 - u_2 = \frac{g'}{f}\partial_y\eta_2, \qquad (7.12)$$

which are called the *thermal wind* relation, as they were first discovered in, and applied to, atmospheric dynamics. They show that in the geostrophic limit the horizontal gradient of the height of the interface is related to the velocity difference across the interface perpendicular to the gradient of the height of the interface.

This finding can of course be generalised to models with several layers and also to the limit of an infinity of layers, that is, to a continuous variation of density and velocity. Which than means in the geostrophic limit: if we know the density structure of the ocean, we know the vertical gradient of the horizontal velocity every where. If we knew the velocity at a certain depth we could use the thermal wind relation to calculate the velocity every where. As the geostrophic velocities in the deep ocean are usually smaller than near the surface, oceanographers conjecture a *level of no motion* which is set rather arbitrarily to, for example, 4000m depths, to calculate the geostrophic velocities every where.

The thermal wind relation was of paramount importance in the past, when it was difficult to measure velocities from a ship at open sea. The density structure on the contrary was much easier to determine precisely. Today with the help of satellites the measurements of velocities have become much more precise, and comparisons with the density structure show the good agreement with the thermal wind relations.

Exercise 53: Where would the velocities be directed in fig. 7.1 on the southern hemisphere?

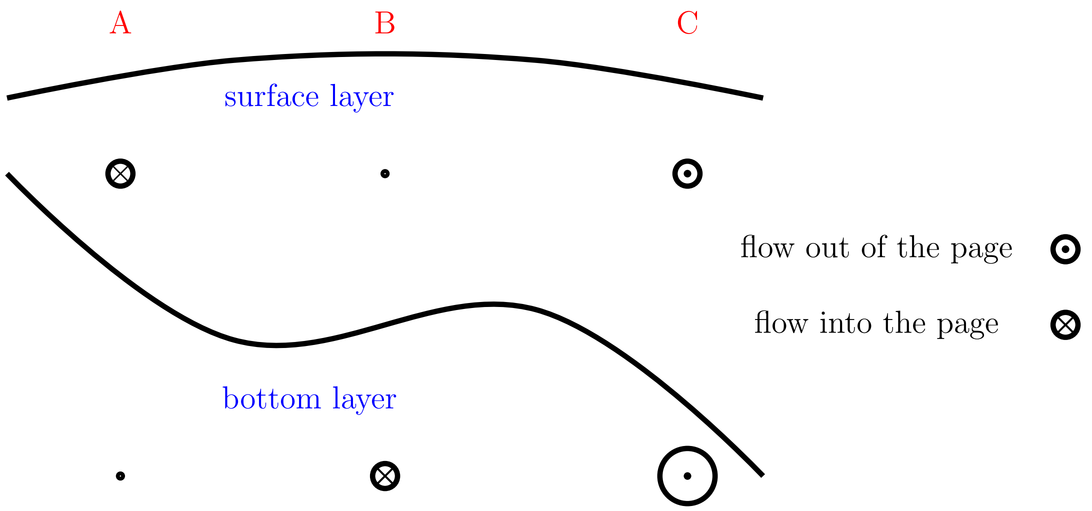


Figure 7.1: Geostrophy in a two layer model: in region A the pressure gradient of the inter-facial slope compensates the pressure gradient of the surface slope and the lower layer is inert; in region B the surface is level and there is no geostrophic motion in the surface layer, the inter-facial slope corresponds to a velocity in the bottom layer; in region C the slope of the surface and the interface lead to a higher velocity in the bottom layer. The slope of the surface is exaggerated with respect to the interface slope, the vertical variations of the interface are of the order of 1m, while the interface varies hundreds of meters. The situation presented corresponds to the northern hemisphere.

7.4 Barotropic versus Baroclinic

Barotropic flow means that iso-barique surfaces coincide with iso-density surfaces. This is the case if and only if $\eta_1 = 0$ for all x, y in the two layer case or $\eta_i = 0$ for all x, y and for all $i < N$ in the multi-layer case (index are counted from top to bottom, 1 being the surface layer and N the bottom layer). Geostrophy applied to (7.1) – (7.6) and the fact that $g' + g'' = g$, we see that the geostrophic flow does not change with depth which is the case for purely barotropic flow. In oceanography the barotropic component of the flow is a component for which the horizontal velocity does not change in the vertical direction. Confusion often arises because sometimes the depth average velocity is called the barotropic component and sometimes it is the geostrophic flow corresponding to the surface elevation ($\sum_{i=1}^{N} \eta_i$). Anyway, the differences between the actual flow and the barotropic flow is called the baroclinic flow. So there is vertical shear in the horizontal velocity field if and only if the baroclinic flow is not vanishing.

Exercise 54: Give an example to show that the two definitions of " barotropic component" differ.

7.5 Eddies, Baroclinic instability (qualitative)

There is one important phenomena that we can not explain from what we have learned so far and this is the abundance of oceanic eddies with the size of approximately the first baroclinic Rossby radius, that is around 100km. The maximum speed in these eddies is 1ms^{-1}. Indeed when the first satellite observations of the ocean where available the ocean looked like a "sea of eddies", a feature that is well reproduced by today's numerical models of the ocean circulation.

Observations and numerical simulations show that at many locations in the ocean the velocity fluctuations due to eddying motion are up to two orders of magnitude larger than the average velocity.

Exercise 55: Search the Internet for maps of sea surface height (SSH) from observations and numerical models. Where do you see the eddies?

Exercise 56: Estimate the vorticity ζ of an ocean eddy and its Rossby number $Ro = \zeta/f$. Are eddies well described close to a geostrophic equilibrium?

Comment: The *Rossby number* and the baroclinic *Rossby radius* are two different things with no direct connection, they are just named after the same person. The Rossby number compares the relative vorticity to the planetary vorticity, or the magnitude of the non-linear term to the Coriolis term. While the Rossby radius is the distance a gravity wave of speed $\sqrt{g'H}$ has traveled in the time f^{-1}.

Exercise 57: Estimate the SSH anomaly at the eddy center of a Gulf Stream eddy.

The process of formation of these eddies, which is called *Baroclinic instability*, is not only an oceanic phenomenon but the cyclones and anticyclones in the mid-latitudes which determine our weather are their atmospheric counterparts and are dynamically the same process. In the atmosphere the baroclinic Rossby radius is of the order of 1000km which explains the size of the cyclones and anticyclones in the atmosphere. It is clear that these "eddies" are key to our understanding of the atmospheric dynamics but in the ocean they are rather small, do they affect the large scale ocean dynamics? YES! they do! We have seen in section 5.9 that geostrophic dynamics at scales larger than the baroclinic Rossby radius has most of its energy stored as *availabel potential energy* which is constantly supplied by the wind-stress through Ekman pumping at a scale which is roughly 30 times the baroclinic Rossby radius.

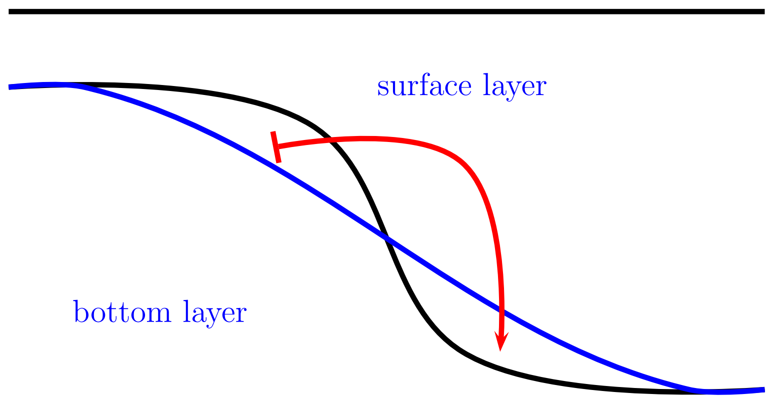


Figure 7.2: Baroclinic instability flattens the interfacial surface. The black line represents the surface before and the blue line after the baroclinic instability. This leads to a downward transport of heavy fluid, that is, a release of available potential energy, as indicated by the red arrow. It does so by forming eddies which mixes the interfacial layer thickness.

The available potential energy is thus $30^2 \approx 1000$ larger than the kinetic part, as the flow is close to a geostrophic equilibrium. This energy has to be drained somehow. That is what the baroclinic instability does by generating eddies at the scale of the baroclinic Rossby radius. We see that the energetics of the large scale circulation can not be understood without this important process. We remind the reader that in chapter 6 we did only consider the conservation of potential vorticity but did not mention energy.

The eddies, themselves not being far from geostrophy, contain about the same amount of available potential energy and kinetic energy as they are of the size of the baroclinic Rossby radius. So baroclinic instability transfers large scale available potential energy to small scale kinetic and available potential energy. Where does the energy go from there? Eddies interact form smaller and smaller structures as, for example, filaments which are then dissipated away in a turbulent cascade process. The merger of eddies also forms lager structures, leading to an energy transport back to larger scales. Eddies also transport the water masses in the latitudinal direction and lose their temperature anomalies by surface fluxes. That is, for example: warm core Gulf Stream eddies travel north loose, their heat to the atmosphere and fade away, transporting substantial parts of heat from low to high latitudes.

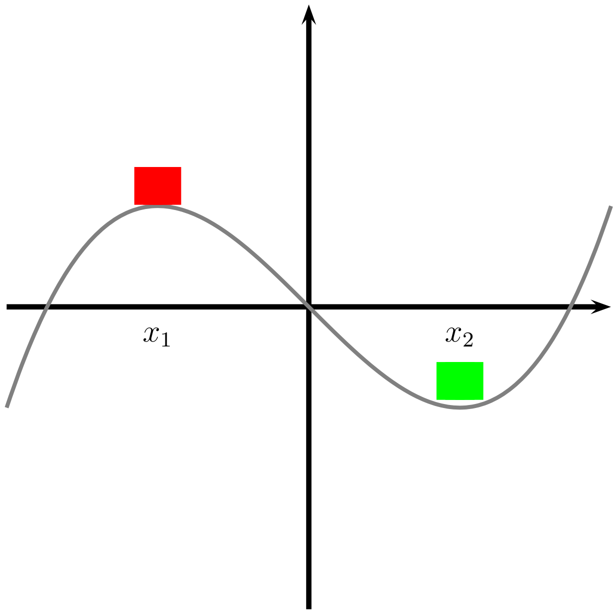


Figure 7.3: Two blocks on a surface at stationary points, the green block is stable, the red block is unstable.

7.6 Baroclinic instability (quantitative)

Instability in $1D$

For those who have never calculated the stability of a state, here a simple example from mechanics. Suppose a glider at position x of mass one in a gravitational field on a surface of the form $P(x) = x^3/3 - x$ (see fig. 7.3). The governing equation equation is,

$$\partial_{tt} x = -\partial_x P = 1 - x^2. \tag{7.13}$$

The two stationary solutions are $x_1 = -1$ and $x_2 = 1$. We now suppose that the solutions are slightly perturbed (at $t = 0$) by $x(0) = x_i + \epsilon x'(0)$, with $\epsilon \ll 1$. Equation (7.13) now reads:

$$\epsilon \partial_{tt} x' = -(x_i + \epsilon x')^2 + 1. \tag{7.14}$$

At $O(1)$ the equation is trivial and at $O(\epsilon)$ it is:

$$\partial_{tt} x' = -2 x_i x'. \tag{7.15}$$

which has the solution:

$$x'(t) = x'(0) \exp(\sqrt{-2x_i}\, t). \tag{7.16}$$

This shows that at x_1 the solution is unstable (grows exponentially in time) and for x_2 it performs oscillations with a constant amplitude. This is called "overstability", the restoring force is so strong that the block over-shoots from the stable situation, this is characteristic for a stable state in a non-dissipative system.

Quasi Geostrophy

The basis of the quasi geostrophic (QG) model is the conservation of potential vorticity:

$$\frac{d}{dt}Q = \frac{d}{dt}\left(\frac{\zeta + f}{H + \eta}\right) = 0. \tag{7.17}$$

The deficiency of the above equation is that we can not rigorously calculate the velocity field, that transports the potential vorticity Q, from the potential vorticity. This so-called inversion can however be done to leading order by using geostrophy:

$$f\tilde{v} = g\partial_x \eta \quad \text{and} \quad f\tilde{u} = -g\partial_y \eta \tag{7.18}$$

then,

$$\tilde{\zeta} = \frac{g}{f}\nabla^2 \eta \tag{7.19}$$

If we define the geostrophic stream function $\Psi = (g/f)\eta$, the potential vorticity can then be replaced (to leading order) by the quasi geostrophic potential vorticity:

$$\tilde{Q}H = \nabla^2 \Psi - \frac{f^2}{gH}\Psi \tag{7.20}$$

Exercise 58: Show how eq. (7.20) can be obtained asymptotically from eq. (7.17). (Rem: use that $Fr^2\epsilon^{-1} \ll 1$ and $\epsilon \ll 1$ with the Froud number $Fr = u/(gh)$ and the Rossby number $\epsilon = u/(Lf)$, u being a typical velocity scale and l a typical horizontal length scale.

The conservation law then is:

$$(\partial_t + \tilde{u}\partial_x + \tilde{v}\partial_y)\tilde{Q} = (\partial_t + \tilde{u}\partial_x + \tilde{v}\partial_y)(\nabla^2\Psi - \frac{f^2}{gH}\Psi) = 0. \tag{7.21}$$

Please note that $r^2 = gH/f^2 = k_r^{-2}$ is the square of the Rossby radius of deformation.

For a two layer model with layers of equal thickness H and a reduced gravity of g', the quasi geostrophic conservation of potential vorticity is:

$$(\partial_t + \tilde{u_1}\partial_x + \tilde{v_1}\partial_y)(\nabla^2\Psi_1 + \frac{f^2}{g'H}(\Psi_2 - \Psi_1)) = 0. \tag{7.22}$$

$$(\partial_t + \tilde{u_2}\partial_x + \tilde{v_2}\partial_y)(\nabla^2\Psi_2 + \frac{f^2}{g'H}(\Psi_1 - \Psi_2)) = 0. \tag{7.23}$$

Where we used the curl of the thermal wind relation (eqs. 7.11 7.12) and that $g' \ll g$, which says that the variation of the layer thickness is essentaily due to the baroclinic mode. In the sequal I will drop the $\tilde{.}$ as it is always clear when I refere to the QG-PV.

Exercise 59: Derive eqs. (7.22) and (7.23).

BC in a Two Layer Model (Phillips Problem)

I will consider the dynamics of the baroclinic instability in a two layer QG model, this is usally refered to as the *Phillips problem*.

We start with a basic zonal and stationary two-layer flow that has a vertical shear and is constant in the horizontal (no x or y dependence) $U_1 = -U_2 = U$. Linearizing equations (7.22) and (7.23) around the basic flow and denoting the perturbations by a prime we obtain:

$$(\partial_t + U\partial_x)\left(\nabla^2\Psi_1' + k_r^2(\Psi_2' - \Psi_1')\right) + 2k_r^2 U\partial_x\Psi_1' = 0. \tag{7.24}$$
$$(\partial_t - U\partial_x)\left(\nabla^2\Psi_2' + k_r^2(\Psi_1' - \Psi_2')\right) - 2k_r^2 U\partial_x\Psi_2' = 0. \tag{7.25}$$

Every solution of this linear equation in Ψ_i' can be written as a sum of exponential modes (eigen-functions):

$$\Psi_i' = Re\left(\tilde{\Psi}_i \exp i(kx + ly - \omega t)\right) \tag{7.26}$$

substituting this into eqs. (7.24) and (7.25) we get:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tilde{\Psi}_1 \\ \tilde{\Psi}_2 \end{pmatrix} = 0 \tag{7.27}$$

with:

$$\begin{aligned} a_{11} &= -c(K^2 + k_r^2) + U(K^2 - k_r^2) && (7.28)\\ a_{12} &= -(U - c)k_r^2 && (7.29)\\ a_{21} &= -(U + c)k_r^2 && (7.30)\\ a_{22} &= c(K^2 + k_r^2) + U(K^2 - k_r^2), && (7.31) \end{aligned}$$

where $c = \omega/k$ and $K^2 = k^2 + l^2$. If there exists a non-trivial solution the determinant $a_{11}a_{22} - a_{12}a_{21}$ has to vanish which is equivalent to:

$$\begin{aligned} 0 &= \left[-c(K^2 + k_r^2) + U(K^2 - k_r^2)\right]\left[c(K^2 + k_r^2) + U(K^2 - k_r^2)\right] - (U^2 - c^2)k_r^4 \\ &= U^2(K^2 - k_r^2)^2 - c^2(K^2 + k_r^2) - (U^2 - c^2)k_r^4 \\ &= U^2K^2(K^2 - 2k_r^2)^2 - c^2K^2(K^2 + 2k_r^2) \end{aligned} \tag{7.32}$$

and we have:

$$c = \pm U\sqrt{\frac{K^2 - 2k_r^2}{K^2 + 2k_r^2}}. \tag{7.33}$$

For $K^2 < 2k_r^2$ the solution is imaginary and a perturbation grows exponentially, it is linearly unstable.

Please note that:

- There is instability for all values of U (this changes when the β-effect is included).
- There is a high-wavenumber cut-off ($K^2 < 2k_r^2$)
- There is no low-wavenumber cut-off (this changes when the β-effect is included).
- When the same calculations are performed for a β-plane (more involved) there are three important differences: (i) it introduces anistropy in the x-y-plane, (ii) there is a low wave number cut-off, that is the large scale perturbations are stabalized, (iii) there is a threshold for the shear, below which the flow is stable.

Exercise 60: For what value of K do we have the fastes growth rate?

7.7 Continuous Stratification

Observations of the ocean indicate, that there are over substantial parts of the ocean areas where the water mass properties and the velocities are almost constant in the vertical direction, separated by sudden jumps in these variables. So the ocean is often well described by layers and this is the basis of the success of layered models. Dividing the ocean in more and more layers that is $\lim N \to \infty$ one approaches a continuous stratification.

Chapter 8

Equatorial Dynamics

The ocean dynamics near the equator is different from other places on our planet as the Coriolis parameter $f = 2\Omega \sin\theta$, measuring the vertical component of the rotation vector, a key parameter in geophysical fluid dynamics, vanishes at the equator. We remind the reader that the ocean currents are mostly horizontal and we can thus to first order neglect the horizontal component of the rotation vector. The terms containing the horizontal component of the rotation vector always involve the vertical component of the velocity vector due to the orthogonal nature of the vector product $\mathbf{\Omega} \times \mathbf{u}$. Neglecting the horizontal component of the rotation vector is called the *traditional approximation* . This does not mean that the effects of rotation can be neglected when considering equatorial dynamics. Although the Coriolis parameter vanishes at the equator its change with the respect to the meridional direction, $\beta = 2(\Omega/R)\cos\theta$, where R is the earths radius, is maximal at the equator. The equatorial dynamics is thus well described by what is called the *equatorial β-plane.* The reduced gravity shallow water equations for the equatorial β-plane are given by eqs. (5.45) - (5.47) with $f = \beta y$.

Another peculiarity of equatorial dynamics is the strong density stratification across the thermocline. At the equator radiative forcing is strongest leading to warm waters and there is also no cooling of the surface waters in winter time, a process important at high latitudes. Precipitation is also strong near the equator freshening the surface waters. Both phenomena lead to strong vertical density differences at low latitudes, which are responsible for a strong vertical shear of the horizontal velocity.

The first question we have to address is of course about the latitudinal extension of the equatorial β-plane. If we compare the wave speed $c = \sqrt{g'H}$ to the value of β we obtain the equatorial Rossby radius $R_{eq} = \sqrt{c/\beta}$. For barotropic dynamics $H \approx$ 4km we obtain $R_{eq} \approx$ 3000km. Due to the strong vertical density difference across the equatorial thermocline most phenomena are, however, baroclinic in the tropics (at low latitude). For such dynamics g has to be replaced by the reduced gravity $g' = g\Delta\rho/\rho$ and the relevant thickness is this of the layer above the thermocline. For this reduced gravity dynamics of the waters above the thermocline $c^{bc} = 0.5\text{ms}^{-1}$ which leads to $R_{eq}^{bc} \approx$ 300km. This gives a band extending approximately 3^o to the north and south of the equator, a rather large area.

The easterly winds (winds coming from east) drive the westward (to the west) *equatorial current* . These current causes a pileup of water at the western side of the basin, which leads to a eastward *equatorial undercurrent* just below the waters directly influenced by the windstress. The equatorial undercurrent is a band of eastward moving water at about 200m depth which is about 100m thick and 300km large and which has maximal velocities of up to 1.5 ms^{-1} in the Pacific Ocean. The equatorial pile up of water at western side of the basin also leads to eastward (counter) currents at the surface north and south of the equator, which are called *north equatorial counter current* (NECC) and *south equatorial counter current* (SECC), respectively. Due to the north south asymmetry of the wind forcing, the NECC is usually more pronounced than the SECC, which is often not observed. These currents exist in all three ocean basins, but their exact location and strength differs. In the Indian Ocean these currents reverse due to the reversing monsoon wind forcing.

Chapter 9

Abyssal and Overturning Circulation

The study of the deep circulation of the world ocean has historically relied on the analysis of water masses. The reasons are that: (i) in the deep water masses change very slowly in time as they are not subject to boundary forcing and as they give an integrated view of the velocity field which mostly weakens when descending into the depth of the ocean; (ii) it is technically difficult to measure the moderate but highly variable velocities in the deep ocean, especially from a ship that is transported by the stronger currents at the ocean surface.

In 1751 Stephen Hales constructed a "bucket sea-gage" and asked Henry Ellis, the captain of the *Earl of Halifax*, to perform temperature measurements in the deep North Atlantic. Ellis found that temperature decreases with depth and noted: "This experiment, which seemed at first but mere food for curiosity, became in the interim very useful to us. By this means we supplied our cold bath, and cooled our wines or water at pleasure; which is vastly agreeable to us in this burning climate."

It was Count Rumford who noted in 1800 that this cold water can only originate from high latitudes and called it "[...] an inconvertible proof of the existing of cold water at the bottom of the sea, setting from the poles towards the equator." This picture was then refined and the zones of formation of the deep waters were identified to lie in the high latitudes of the North Atlantic and the Antarctic Ocean. There is no formation of deep waters in the Indian and Pacific Ocean. The deep waters are upwelling in the rest of the ocean counter balancing the diffusion of heat into the deep ocean and thus forming the thermocline , that is a more or less sharp boundary between the warm surface waters and the cold deep waters in the mid and low latitudes. These processes are schematised in fig. 9.1.

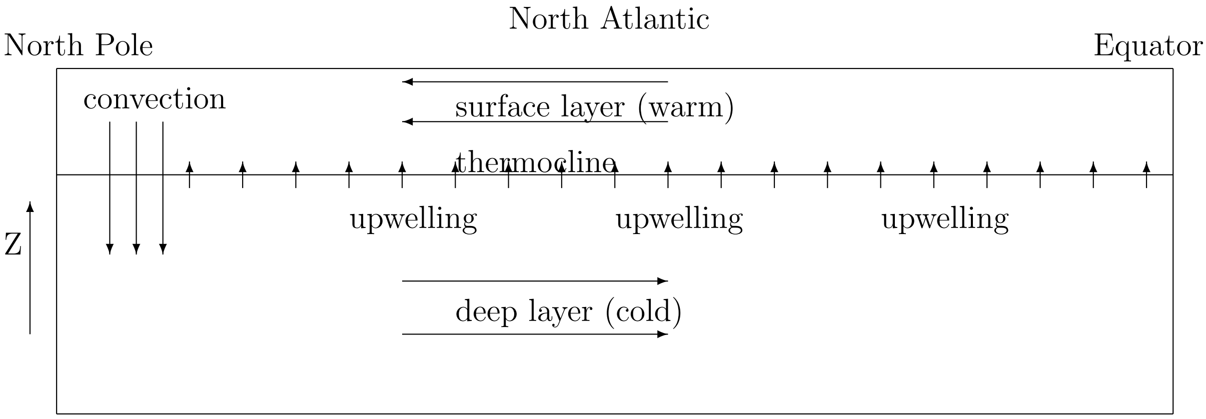


Figure 9.1: Overturning Circulation

Only in the beginning of the 20th century Chamberlain (1906) considered the possibility of variability or even a reversal of the deep ocean currents and its effects on climate. The vulnerability of the overturning circulation to changes in the freshwater forcing at the ocean surface is today seen as a likely candidat for the abrupt (several decades) *Dansgaard-Oeschger* climate change events.

9.1 The Stommel Arons Theory

The picture presented in the previous section led Stommel and Aarons to consider the dynamics of the deep layer in the ocean, which is subject to a localized injection of water in the northern part and an upwelling, from the deep layer into the surface layer, through out the rest of the thermocline. In the simplest geometry our ocean is a slice of the earth confined between longitudes ϕ_w, ϕ_e and in the south by the equator. The injection happens at the North Pole and has a strength S_N (measured in Sv). As the total volume of the deep layer is conserved the upwelling velocity $w_{up} = S/A$ where A is the surface of our slice, $A = R^2(\phi_e - \phi_w)$. This positive vertical velocity leads to a stretching of the deep layer and thus by conservation of potential vorticity (eq. 5.70 or actually planetary potential vorticity) to a northward velocity which is given by $v_{sv} = f w_{up}/(H\beta) = w_{up} R \tan(\theta)/H$. But north-ward velocity means towards the source! Conservation of mass imposes a southward transport somewhere in the fluid, and knowing Sverdrup theory we suspect this transport to occur on the western boundary.

North Atlantic

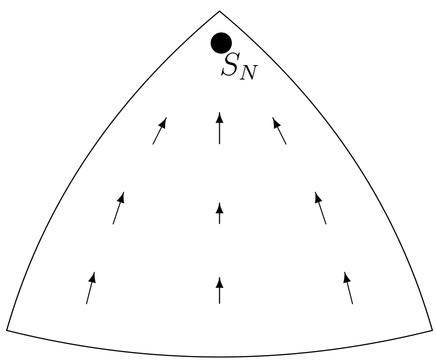


Figure 9.2: Stommel-Arons Model

The north-ward interior (Sverdrup) transport as a function of latitude is

$$T_{sv} = v_{sv} H(\phi_e - \phi_w) R cos(\theta) = w_{up} R^2 (\phi_e - \phi_w) \sin(\theta) = S_N \sin(\theta). \tag{9.1}$$

The vertical transport into the deep layer north of the latitude θ (it actually goes out of the deep layer it has a minus sign!) is equal to minus the upward velocity times the surface,

$$T_{up} = -w_{up} R^2 (\phi_e - \phi_w) \int_{\theta}^{\pi/2} cos(\theta') d\theta' = -w_{up} R^2 (\phi_e - \phi_w)(1 - \sin(\theta)) = S_N(\sin(\theta) - 1). \tag{9.2}$$

For a slice north of θ we have,

$$S_N + T_{sv} + T_b + T_{up} = 0 \tag{9.3}$$

Which allows us to calculate the boundary transport:

$$T_b = -2S_N \sin(\theta), \tag{9.4}$$

this means, that at the apex the boundary current has twice the strength of the source.

There is clear evidence of the existence of an overturning circulation in the temperature structure of the worlds ocean, as stated in the beginning of this section. The overturning circulation itself is, however, very difficult to observe as the velocities v_{sv} and w_{up} are small and such difficult to measure. The convection process is very localised in time and space and difficult to quantify. The boundary current such seems to be the only easily observable branch of the overturning circulation! All the branches of the overturning circulation are clearly observed in today's numerical models of the ocean circulation.

Exercise 61: What happens when the strength of the source is increased or decreased?

Exercise 62: What happens when the source is displaced southward?

Exercise 63: What happens when there are more sources?

Exercise 64: What happens when the source is at the equator?

Exercise 65: What happens when the ocean spans a slice from the North- to the South-Pole? What happens at the equator?

9.2 Multiple Equilibria of the Thermohaline Circulation

The here presented model was introduced by Stommel-Marotzke-Stocker. It represents the most simple model of the thermohaline circulation. There are only two boxes which are characterized by their respective temperature and salinity. The temperature in both boxes is held fix, while the salinity depends on precipitation, this is called: "mixed boundary conditions." Such boundary conditions are reasonable as sea surface temperature (SST) anomalies are damped by heat fluxes, whereas sea surface salinity (SSS) anomalies have no essential influence on precipitation or evaporation.

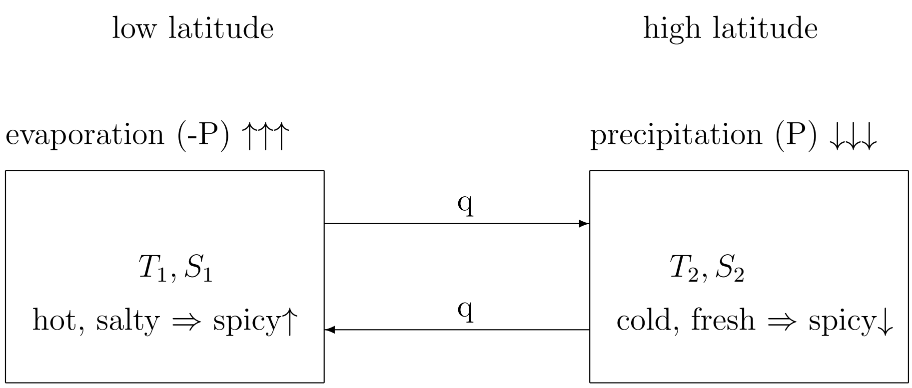


Figure 9.3: Stommel Box-Model (note: low latitude to right)

$$\dot{S}_1 = |q|(S_2 - S_1) + P \tag{9.5}$$

$$\dot{S}_2 = |q|(S_1 - S_2) - P \tag{9.6}$$

$$q = k\alpha(T_2 - T_1) + k\beta(S_2 - S_1) \tag{9.7}$$

Where $\alpha > 0$ and $\beta < 0$ are the expansion coefficients of temperature and salinity, respectively, and k is a coefficient that connects q to the density difference, and $P > 0$. For simplicity of the mathematics, and as we are only interested in qualitative results, we fix $\alpha = 1$, $\beta = -1$ and $k = -1$. The actual values can be adjusted based on observations. We then define $\Delta S = S_2 - S_1$ and $\Delta T = T_2 - T_1$ and note that $\Delta T < 0$ (and $\Delta S < 0$ if $P > 0$)!

$$\frac{1}{2}\Delta\dot{S} = -|q|\Delta S - P, \tag{9.8}$$

with $q = \Delta S - \Delta T$. Looking for stationary states ($\Delta\dot{S} = 0$) we obtain:

$$|\Delta S - \Delta T|\Delta S + P = 0. \tag{9.9}$$

We will call a THC with $q > 0$ forward and with $q < 0$ reverse. Solving these equations we obtain the following stationary states:

$$\Delta S = \frac{1}{2}(\Delta T \pm \sqrt{(\Delta T)^2 - 4P}) \text{ if } q > 0 \tag{9.10}$$

$$\Delta S = \frac{1}{2}(\Delta T - \sqrt{(\Delta T)^2 + 4P}) \text{ if } q < 0 \tag{9.11}$$

A fourth solution contradicts the $q < 0$ condition. We can now distinguish several cases (see also fig. 9.2):

(1) for $P < 0$ an unrealistic forcing, there is only one solution which is a strong forward THC, as salinity and temperature favor a positive q.

(2) $0 < P < (\Delta T)^2/4$ and we have three solutions, one unstable and two stable. The two stable solutions are

$$\Delta S = \frac{1}{2}(\Delta T + \sqrt{(\Delta T)^2 - 4P}) \text{ and } q = \frac{1}{2}(-\Delta T + \sqrt{(\Delta T)^2 - 4P}) > 0 \tag{9.12}$$

$$\Delta S = \frac{1}{2}(\Delta T - \sqrt{(\Delta T)^2 + 4P}) \text{ and } q = \frac{1}{2}(-\Delta T - \sqrt{(\Delta T)^2 + 4P}) < 0 \tag{9.13}$$

What is the physics of these two stationary solutions? The first is the usual fast and forward thermohaline circulation, this means that the THC is so fast that precipitation has no time to act and temperature effects dominate over salinity. The second solution is slower and reversed, the circulation is slow so precipitation can do its job and salinity dominates temperature differences.

(3) $P > (\Delta T)^2/4$ that is strong precipitation and we have only one stationary solution which is dominated by salinity and is an inverse THC (perhaps the Pacific Ocean and the North Atlantic at the end of glacial periods).

We have thus seen that using mixed boundary conditions for temperature and salinity, we can have two solutions for the same forcing! A nonlinear equation can have several solutions for the same set of parameters and boundary conditions.

Another important point is that such ocean model exhibits a hysteresis behaviour as a function of a control variable as for example the precipitation. When small perturbations are added, such model can give rise to abrupt changes between the two stable states followed by periods of stability of arbitrary length. The observed break down of the thermohaline circulation in the North Atlantic is often explained by such kind of model and multiple equilibria.

Exercise 66: We have written all the equations in non-dimensional form. Perform the calculations for a concrete example (for example: volume of the boxes 1m^3, ...).

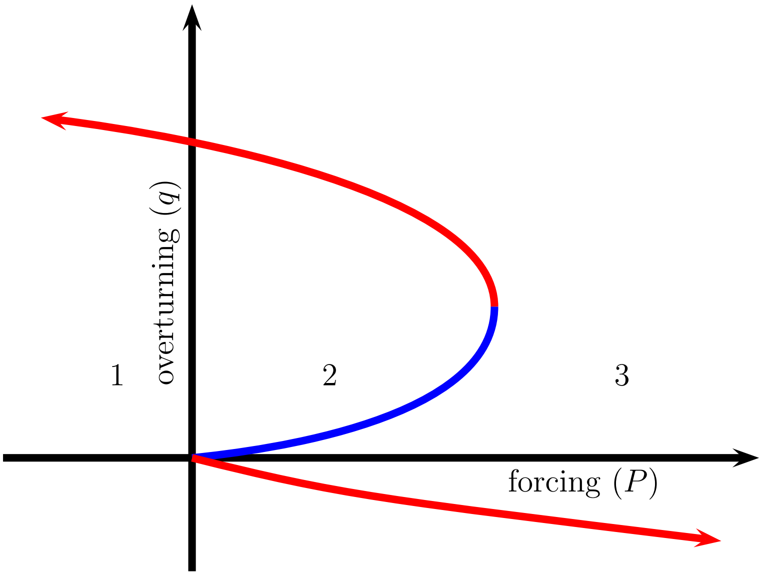


Figure 9.4: Hysteresis behaviour of the model as described by eqs. (9.12) and (9.13). The strength of the overturning circulation (q) is described as a function of the precipitation (P) for a fixed temperature difference (ΔT). Stable solutions are sown in red, the unstable solution in blue. There is one solution in the region 1 and 3. There are two stable solutions and one unstable solution in region 2.

9.3 What Drives the Thermohaline Circulation?

A key question we have not considered so far is, where does the mechanical energy come from that drives the thermohaline circulation and transports the heat? A question we did not consider when discussing the Stommel-Aarons model, which is based on conservation of potential vorticity. The evaporation takes water from the surface which is then, at a different location, reintroduced in the ocean, by rain and river runoff. The important point is, however, that the mass is taken and put back at the ocean surface, that is at the same geopontential height! Which means that no net potential energy (mgh) is provided to the ocean as neither mass nor height is different at evaporation and precipitation points (see fig. 9.3). What about the mechanical work ($dW = -pdV$) done on the ocean by thermal expansion and contraction, that is change of volume dV. Again, both processes happen at the surface, at the same pressure p, and again: no net energy is provided to the ocean by thermal atmospheric forcing. Please note that the situation is completely different for the atmosphere, as shown in fig. 9.3, which is generally heated at a lower geopotential height, typically at the surface, than at which it is cooled, by rayonating energy into space, and mechanical work is provided.

So what drives the THC? Originally it was thought that the forcing comes from the cooled water pushing the thermohaline circulation until Sandström, in 1908, asked the question about the energy balance discussed above. Sandström concluded that in a fluid heated and cooled at the surface the fluid below the cold source, should be homogeneous at the cold temperature and the fluid between the cold and warm sources would be stably stratified with only low fluid velocities. A result that bears the name of Sandström's Theorem.

Then the idea was put forward, that the diffusion of heat from the surface into the depth at low latitudes descends the effective heating into the ocean and provides thus for the missing energy to drive the THC, which meant that the THC is pulled rather than pushed. Recent research initiated by Munk & Wunsch in 1998 favors still another idea, which is that the driving force is the wind. This means that the low to high latitude heat flux of 2×10^{15}W is a passive consequence of the wind driven circulation powered by only 2×10^{12}W, a thousand times less!

The picture they propose is, that the wind stress drives a conveyor belt that transports the heat.

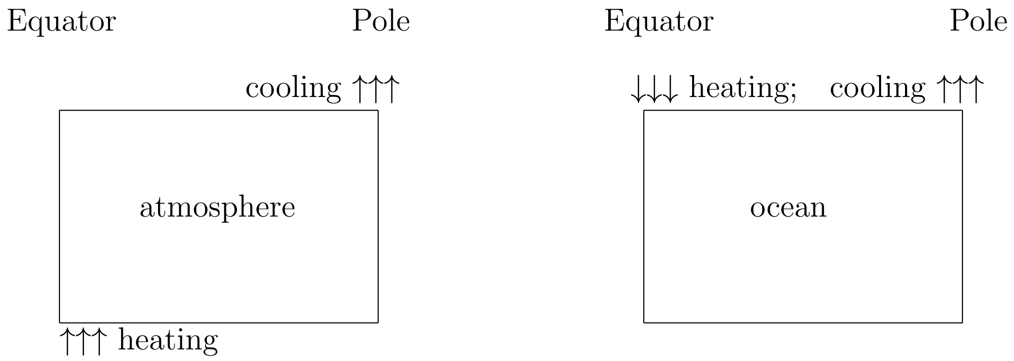


Figure 9.5: Energy Balance

Chapter 10

Penetration of Surface Fluxes

The ocean is mostly driven by the fluxes of heat, fresh water and momentum at its surface. The influence of these fluxes are, however, not only felt in a thin layer at the ocean surface, but influence the dynamics of the entire ocean. In this chapter we discuss how the forcing applied at the surface of the ocean penetrates into the depth of the ocean.

For the processes of vertical penetration, it is clear that we can no longer neglect the dynamics in the vertical direction, and the shallow water equations are not adapted for the processes studied here (with the exception of gravity currents). We thus have to look for other simplifications of the full three-dimensional Navier-Stokes equations. A first guess might be to neglect the dynamics all together and pretend that the transport to the interior is due to molecular motions, that is viscosity and diffusivities (for heat and salt). This possibility is discussed and refuted in section 10.1.

In section 10.2 we show, using the Navier-Stokes equations and some "hand-waving" that the three dimensional dynamics at small scales creates some viscous and diffusive behavior at large scales. This idea is the basis of all realistic calculations not only in ocean dynamics but in fluid dynamics in general.

10.1 Molecular Transport

The molecular thermal diffusivity of sea water is $\kappa \approx 10^{-7}\mathrm{m}^2\mathrm{s}^{-1}$. The diffusion equation in the vertical is given by,

$$\partial_t T = \partial_z(\kappa \partial_z T). \tag{10.1}$$

We further suppose that there is a periodic heat flux of magnitude Q at the surface (boundary condition), that is:

$$\partial_z T|_{z=0} = \frac{Q}{c_p \rho \kappa} \cos(2\pi t/\tau + \pi/4), \tag{10.2}$$

$$\lim_{z \to -\infty} \partial_z T = 0. \tag{10.3}$$

The linear equation (10.1) with the boundary conditions (10.3) has the solution:

$$T(z,t) = T_A e^{z/L} \cos(2\pi t/\tau + z/L), \tag{10.4}$$

with:

$$T_A = \frac{Q}{c_p \rho} \sqrt{\frac{\tau}{2\pi\kappa}} \text{ and } L = \sqrt{\frac{\tau\kappa}{\pi}}. \tag{10.5}$$

Where $Q \approx 200\text{Wm}^{-2}$, $c_p = 4000\text{JK}^{-1}\text{kg}^{-1}$, $\rho = 1000\text{kgm}^{-3}$ if we take τ to be one day we get: $T_A \approx 20\text{K}$ and $L = 5.2\text{cm}$, this means that the surface temperature in the ocean varies by 40K in one day and the heat only penetrates a few centimeters. If τ is one year, considering the seasonal cycle, $T_A \approx 400\text{K}$ and $L \approx 1\text{m}$. This means that the surface temperature in the ocean varies by 800K in one year and the heat only penetrates about one meter. This does not at all correspond to observation!

When using the molecular viscosity, ν we can also calculate the thickness of the Ekman layer $\delta = \sqrt{2\nu/f}$ which is found to be a few centimeters. The observed thickness of the Ekman layer in the ocean is however over 100 times larger.

This shows that molecular diffusion can not explain the vertical heat transport, and molecular viscosity can not explain the vertical transport of momentum! But what else can?

10.2 Turbulent Transport

In the early 20th century fluid dynamicists as L. Prandtl suggested that small scale turbulent motion mixes scalars and momentum very much like the molecular motion does, only that the turbulent mixing coefficients are many orders of magnitude larger than their molecular counterparts. This is actually something that can easily be verified by gently poring a little milk into a mug of coffee. Without stirring the coffee will be cold before the milk has spread evenly in the mug, with a little stirring the coffee and milk are mixed in less than a second.

In this section we like to have a quantitative look at the concept of eddy viscosity in a very simplified frame work that nevertheless contains all the important pieces. The starting point of our investigation are the Navier-Stokes equations (5.1 –5.4). We start by considering the two dimensional motion in the $x-z$-plane. Motion and dependence in the y direction are neglected only to simplify the algebra, and do not lead to important changes. We further suppose that the large scale velocity field is only directed in the x-direction and depends only on the z-direction $U(z)$. The x and z component of the small scale turbulent motion is given by u' and w', respectively.

$$\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} U(z) + u' \\ w' \end{pmatrix} \tag{10.6}$$

with $U(z) = \langle u \rangle_x$.

The $\langle . \rangle_x$ operator denotes the average over a horizontal slice:

$$A(z) = \langle a(x,z) \rangle_x = \frac{1}{L} \int_L a(x,z) dx \tag{10.7}$$

In the sequel we will use the following rules:

$$\langle \lambda(z) a \rangle_x = \lambda(z) \langle a \rangle_x \tag{10.8}$$

$$\langle \partial_z a \rangle_x = \partial_z \langle a \rangle_x \tag{10.9}$$

and

$$\langle \partial_x a(x,z) \rangle_x = \frac{1}{x2 - x1} \int_{x1}^{x2} \partial_x a(x,z) dx = \frac{a(x_2,z) - a(x_1,z)}{x_2 - x1}, \tag{10.10}$$

which vanishes if $a(x,z)$ is bounded and we take the limit of the averaging interval $L = (x2 - x1) \to \infty$. But of course:

$$\langle ab \rangle_x \neq \langle a \rangle_x \langle b \rangle_x . \tag{10.11}$$

If we suppose $u' = w' = 0$, that is, no turbulence the Navier-Stokes equations (see eq. (5.1) – (5.4)) become:

$$\partial_t U + F = \nu \partial_{zz} U \tag{10.12}$$

where $F = (\partial_x P)/\rho$ is the pressure Force. If we allow for small scale turbulent motion we get:

$$\begin{aligned} \partial_t(U + u') + \partial_x\left((U + u')(U + u')\right) + \partial_z\left(w'(U + u')\right) \\ +F + \partial_x p' = \nu \partial_{zz}(U + u'). \end{aligned} \tag{10.13}$$

Where we have used the identity

$$u\partial_x u + w\partial_z u = \partial_x(uu) + \partial_z(uw), \tag{10.14}$$

which is a direct consequence of the incompressibility ($\partial_x u + \partial_z w = 0$). Applying the horizontal averaging operator to eq. (10.13) we get:

$$\partial_t U + \partial_x \langle (U + u')(U + u') \rangle_x + \partial_z \langle w'(U + u') \rangle_x + F + \langle \partial_x p' \rangle_x = \nu(\partial_{xx} + \partial_{zz}) \langle (U + u') \rangle_x \tag{10.15}$$

which simplifies to:

$$\partial_t U + \partial_z \langle w'u' \rangle_x + F = \nu \partial_{zz} U \tag{10.16}$$

If we now compare eqs. (10.12) and (10.16) we see that the small scale turbulent motion adds one term to the large scale equations. The value of this term depends on the small scale turbulence and the large scale flow and is usually unknown. There are now different ways to parametrise this term, that is, express it by means of the large scale flow. The problem of finding a parametrisation is called *closure problem*.

None of the parametrisations employed today is rigorously derived from the underlying Navier-Stokes equations, they all involve some "hand-waving." We will here only discuss the simplest closure, the so called *K-closure* (also called *Boussinesq assumption*).

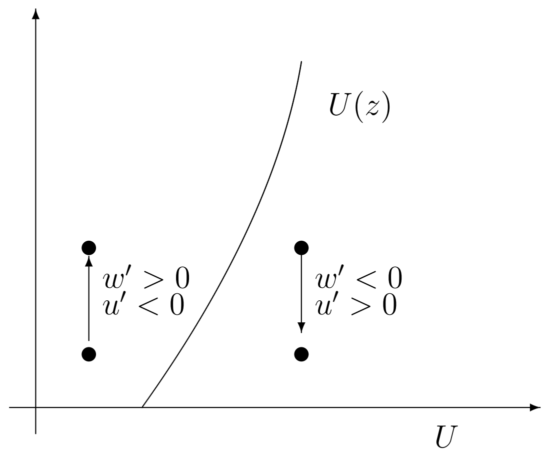


Figure 10.1: K-closure

The K-closure assumes the turbulent flux term to be proportional to the large-scale velocity gradient:

$$\langle w'u' \rangle_x = -\nu'_{eddy} \partial_z U \tag{10.17}$$

where $-\nu_{eddy}$ is the proportionality coefficient. Looking at fig. 10.1 this choice seems reasonable: firstly the coefficient should be negative as upward moving fluid transport a fluid parcel that originates from an area with a lower average velocity in the x-direction to an area with a higher average velocity in the x-direction, such that u' is likely to be negative. The reverse is true for downward transport. Such that $\langle w'u' \rangle_x$ is likely to be negative. Secondly, a higher gradient is likely to increase $|u'|$ and such also $-\langle w'u' \rangle_x$.

Using the K-closure we obtain:

$$\partial_t U + F = (\nu + \nu'_{eddy})\partial_{zz} U, \tag{10.18}$$

which is identical to eq. (10.12) except for the increased effective viscosity $\nu_{eddy} = \nu + \nu'_{eddy}$ called the eddy viscosity

Exercise 67: perform the calculations without neglecting the motion and dependence in the y-direction.

Exercise 68: perform the calculations for a passive scalar (a scalar quantity that diffuses and is transported by the fluid without acting on the velocity field).

10.3 Convection

Oceanic convection is the buoyancy driven vertical mixing of water masses. Convection occurs when the water column is unstable, that is, heavier water is lying above lighter water. In the ocean this typically occurs when the surface waters are either cooled by atmospheric forcing or their salinity is increased by evaporation. (Note that for waters with a salinity above 25PSU density always decreases when temperature increases). If a isothermal ocean of depth H is subject to a heat flux of Q its temperature change is given by:

$$\partial_t T = \frac{Q}{c_p \rho H} \tag{10.19}$$

In extreme cooling events in polar oceans the heat flux can reach $Q = -10^3 \mathrm{Wm}^{-2}$. A typical value of the heat capacity of sea water is $c_p = 4000.\mathrm{Jkg}^{-1}\mathrm{K}^{-1}$.

10.4 Richardson Number

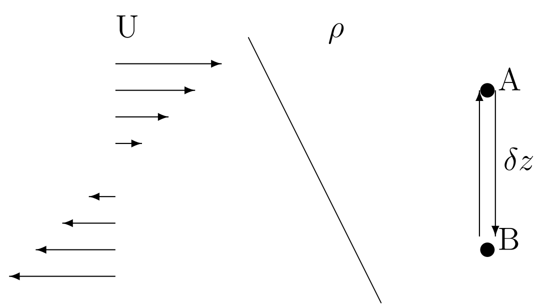


Figure 10.2: Exchanging volumes A and B in a sheared stably stratified flow.

When considering the vertical mixing in the ocean we usually have a large scale horizontal flow that has a vertical shear $\partial_z U$ which has a tendency to destabilize the flow and generate turbulence. On the other hand the flow usually has a stable stratification that suppresses instability and also turbulence. This means that there are two competing phenomena and it is key for vertical mixing to determine under which circumstances one of the processes dominates. To this end we look at a stably stratified sheared flow, and consider the energy budget when to equal volumes A and B, as shown in fig. 10.2, separated by a distance δz are exchanged. The potential energy ΔE_{pot} necessary to exchange the heavier and lower volume B with the lighter and higher volume A is supposed to be provided by the kinetic energy ΔE_{kin} in the shear. For this to be possible it is clear, that $\Delta E_{total} = \Delta E_{kin} + \Delta E_{pot} > 0$ which are given by,

$$\Delta E_{kin} = 2\frac{\rho V}{2}((\delta z/2)\partial_z U)^2 \tag{10.20}$$

$$\Delta E_{pot} = gV(\delta z)^2 \partial_z \rho. \tag{10.21}$$

Note that $\partial_z \rho < 0$ Indeed, $E_{pot} = gh\Delta m$, and for our case $h = \delta z$ and $m = -\delta z V \partial_z \rho$ is the mass difference between volume B and A. $\Delta E_{total} > 0$ if the Richardson number ,

$$Ri = \frac{-g\,\partial_z \rho}{\rho(\partial_z U)^2} < \frac{1}{4}, \tag{10.22}$$

or if we write $\delta U = \delta z\ \partial_z U$ and $\delta \rho = \delta z\ \partial_z \rho$ we obtain,

$$Ri = \frac{-g\,\delta\rho\ \delta z}{\rho(\delta U)^2} < \frac{1}{4}, \tag{10.23}$$

Which means that using the kinetic energy of the volumes A and B it is possible to interchange the volumes A and B when $Ri < 1/4$. Although that this calculation is very simple, only comparing kinetic to potential energy, and does not tell us how the volumes A and B should be exchanged, it is found in laboratory experiments that sheared stratified flow does indeed become unstable around a critical Richardson number of one quarter.

The above, and more involved, calculations together with laboratory experiments and oceanic observations have led to a variety of parametrisations of the vertical mixing based on the Richardson number.

One of the simplest, and widely used, parametrisations for vertical mixing based on the Richardson number was proposed by Philander and Pacanowski (1981):

$$\nu_{eddy} = \frac{\nu_0}{(1+\alpha Ri)^n} + \nu_b, \tag{10.24}$$

where typical values of the parameters, used in today's numerical models of the ocean dynamics, are $\nu_0 = 10^{-2}\mathrm{m^2s^{-1}}$, $\nu_b = 10^{-4}\mathrm{m^2s^{-1}}$, $\alpha = 5$ and $n = 2$.

Exercise 69: *Slippery Sea*

10.5 Entrainment

Entrainment is the mixing of ambient (non or less turbulent) fluid into a turbulent current so that the initially less turbulent fluid becomes part of the turbulent flow. Examples are: a fluid jet that spreads and entrains ambient fluid with it, (ii) an avalanche that entrains surrounding air and increases in size. The fluid flow is typically from the less turbulent fluid to the more turbulent fluid. Entrainment is usually quantified by the entrainment velocity which is the velocity with which the ambient fluid enters into the turbulent jet through the border separating the two fluids. If the entrainment is negative on speaks of detrainment.

10.6 Gravity Currents

Gravity currents are currents that evolve due to their different density with respect to the surrounding water masses. We can thus distinguish buoyant gravity currents and dense gravity currents.

Buoyant gravity currents are lighter than the surrounding and are thus confined to the surface, an example is fresh river water that enters the ocean. Dense gravity currents on the contrary are composed of water heavier than the surrounding and they thus flow along the topography. Important examples are dense currents that pass through straits (Gibraltar, Denmark, ...) and flow down the continental slopes. We will here consider only the case of dense gravity currents.

When a dense gravity current leaves a strait it is deviated to the right by the Coriolis force and flows along the slope of the topography. When we neglect friction, mixing and entrainment (see section 10.5), the parameters determining the dynamics of the gravity current are the reduced gravity $g' = g\Delta\rho/\rho$ the slope α and the Coriolis parameter f.

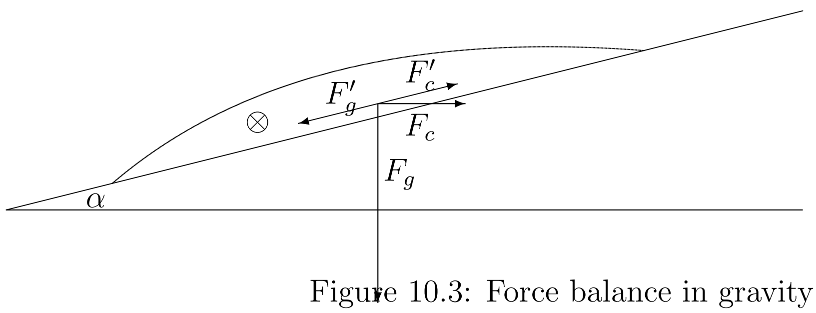


Figure 10.3: Force balance in gravity currents

$$F_g = mg', \qquad F'_g = mg' \sin\alpha, \tag{10.25}$$

$$F_c = mfu, \qquad F'_c = mfu\cos\alpha. \tag{10.26}$$

If we suppose that the gravity current is in a stationary state, the buoyancy force and the Coriolis force projected on the slope have to balance, that is $F'_{g'} = F'_c$ (see Fig. 10.6) and thus,

$$u_{\text{Nof}} = \frac{g'}{f}\tan\alpha, \tag{10.27}$$

which is called the Nof-speed.

Exercise 70: What happens when we include bottom friction in the force balance?

Exercise 71: What happens when we include entrainment in the dynamics (see section 10.5)?

Appendix A

Solution of some exercises

Exercise 12:

Yes, the typical velocity in a Tsunami in deep waters is less than $0.1m/s$ and its horizontal extension is of the order of 100km so the nonlinear term $u\partial_x u < 10^{-7}\text{m}^2\text{s}^{-1}$ much less than $g\partial_x\eta \approx 10^{-4}\text{m}^2\text{s}^{-1}$.

Exercise 13:

The energy of a fluid of density ρ between the two points a and b in a channel of width L is composed of kinetic energy. In the linear shallow water equations they are defined as:

$$E_{kin} = \rho L \int_a^b \frac{H}{2} u^2 dx, \tag{A.1}$$

and potential energy:

$$E_{pot} = \rho L \int_a^b \frac{g}{2} \eta^2 dx. \tag{A.2}$$

The change of the total energy with time is thus:

$$\begin{aligned} \partial_t E_{total} = \partial_t E_{kin} + \partial_t E_{pot} = \frac{\rho L}{2} \int_a^b \left(H\partial_t(u^2)dx + g\partial_t(\eta^2)\right) dx \quad = \\ \rho L \int_a^b \left(-Hgu\partial_x\eta - gH\eta\partial_x u\right) dx = -\rho L g H \int_a^b \partial_x(\eta u) dx \\ = -\rho L g H \left(u(b)\eta(b) - u(a)\eta(a)\right). \end{aligned} \tag{A.3}$$

Where we have used eq. (5.23) and (5.24). So energy is conserved in the domain $[a, b]$ with the exception of energy entering or leaving at the boundary points.

Exercise 14:

$d = \partial_x u + \partial_y v = -\partial_{xy}\Psi + \partial_{yx}\Psi = 0$

Exercise 15:

$\xi = \partial_x v - \partial_y u = \partial_{xx}\Theta + \partial_{yy}\Theta = \nabla^2\Theta$

Exercise 21:

$(x_r, y_r) = R(\cos(\omega t), \sin(\omega t)), \rightsquigarrow (u_r, v_r) = \omega R(-\sin(\omega t), \cos(\omega t)) \rightsquigarrow \partial_t(u_r, v_r) = -\omega^2 R(\cos(\omega t), \sin(\omega t$ putting it together: $((-\omega^2 R - f\omega R - f^2/4R)\cos(\omega t), (-\omega^2 R - f\omega R - f^2/4R)\sin(\omega t)) = 0$, which is satisfied if and only if $\omega = -f/2$

Exercise 17:

now the centrifugal force is balanced by the slope of the free surface and we have $((-\omega^2 R - f\omega R)\cos(\omega t),$ 0 which is satisfied if and only if $\omega = -f$ (compare to previous exercise)!

Exercise 33:

The potential energy released per unit length (in the transverse direction):

$$E_{pot} = 2\frac{1}{2}\rho g \eta_0^2 \int_0^\infty (1 - (1 - \exp(-x/a)))\, dx = \frac{3}{2}\rho g \eta_0^2 a. \tag{A.4}$$

The kinetic energy in the equilibrium (final) solution per unit length (in the transverse direction):

$$E_{kin} = 2\frac{1}{2}\rho H g^2 \eta_0^2 (fa)^{-2} \int_0^\infty \exp(-2x/a) dx = \frac{1}{2}\rho g \eta_0^2 a. \tag{A.5}$$

So energy is NOT conserved during the adjustment process. Indeed waves transport energy from the region where the adjustment occurs to $\pm\infty$.

Exercise 36:

Calculus tells us that:

$$\frac{d}{dt}\left(\frac{\zeta + f}{H + \eta}\right) = \frac{1}{H + \eta}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{(H + \eta)^2}\frac{d}{dt}\eta.$$

So take ∂_x of eq.(5.46) and substract ∂_y of eq. (5.45) to obtain:

$$\begin{aligned} \partial_t \partial_x v \quad &+ \partial_x(u\partial_x v) + \partial_x(v\partial_y v) + f\partial_x u + g\partial_{xy} \quad \eta = 0 \\ -\partial_t \partial_y u \quad &- \partial_y(u\partial_x u) - \partial_y(v\partial_y u) + f\partial_y v - g\partial_{xy} \quad \eta = 0. \end{aligned}$$

After some algebra one obtains:

$$\frac{d}{dt}(\zeta + f) = -(\zeta + f)(\partial_x u + \partial_y v),$$

and eq. (5.47) gives

$$\frac{d}{dt}\eta = -(H + \eta)(\partial_x u + \partial_y v),$$

putting this together gives the conservation of potential vorticity.

Exercise 38:

The moment of inertia is $L = I\omega = mr^2/2\omega = m\rho V/(4\pi)\omega/H$ where we used, that the volume V of a cylinder is given by $V = 2\pi r^2 H$. As the mass and the volume are constant during the stretching of flattening process we obtain that conservation of angular momentum implies that ω/H is constant.

Exercise 44:

The (kinetic) energy is given by: $E = \alpha(U^2 + V^2)$, where the constant $\alpha = A\rho/(2H)$ is the horizontal surface area times the density divided by twice the layer thickness. $\partial_t E = \alpha(\partial_t U^2 + \partial_t V^2) = \alpha 2(U\partial_t U + V\partial_t V)$ using eqs. (6.13) and (6.14) we obtain $\partial_t E = \alpha(fVU + U\tau_x - fUV) = 0$, as the velocity is perpendicular to the forcing, that is, $U = 0$.

Exercise 47:

Summing the first and the third line of the matrix equation gives $U_1 + U_2 = 0$, summing the second and the fourth line of the matrix equation gives $f(V_1 + V_2) = \tau_x$. Eliminating U_2 and V_2 in the first and third equation we obtain:

$$\begin{aligned} \tilde{r}U_1 - fV_1 &= \tau_x \\ fU_1 + \tilde{r}V_1 &= (r/f)\tau_x. \end{aligned}$$

with $\tilde{r} = r(H_1 + H_2)/(H_1 H_2)$ solving this equations give:

$$\begin{aligned} U_1 &= \frac{r\tau_x}{f^2 + \tilde{r}^2}\frac{1}{H_1} \\ V_1 &= -\frac{r\tau_x}{f^2 + \tilde{r}^2}(\frac{f}{r} + \frac{\tilde{r}}{fH_2}) \\ U_2 &= -\frac{r\tau_x}{f^2 + \tilde{r}^2}\frac{1}{H_1} \\ V_2 &= \frac{r\tau_x}{f^2 + \tilde{r}^2}\frac{\tilde{r}}{fH_1} \end{aligned}$$

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